The Sugar Pie Game

Maple Syrup variant of a Sugar pie

We are correct, as it seems to us, in believing that those who stay behind baking a welfare pie for public consumption do not realize that citizens' demands of welfare and public goods to a greater extent might sometimes worsen the quality of cooking the pie in the welfare policy oven. This is how we perceive it and hope that the process of finding the solution in a sugar pie game is the best way to understand what happens. We invite the reader to play the game, which explains the situation with welfare pie in simple terms. The game may be connected to how a piece of sugar pie is fairly shared between two people: HE, a soft negotiator, not very keen on sweets but with emphasis on quality; and SHE, a tough negotiator, likes sweets, whatever they are. The question about the size of the pie we leave temporarily aside for the present.

The axiomatic bargaining theory finds the asymmetric Nash solution by maximizing the product of players' expectations above the disagreement point $d = \langle d_1, d_2 \rangle$:

$$\arg\max_{0 \le x+y \le 1} f(x,y,\alpha) = (u(x) - d_1)^{\alpha} \cdot (g(y) - d_2)^{1-\alpha},$$

the asymmetric variant (Kalai [1977]). [1].

Although the answer may be known to game theory purists, the questions often asked by many include the following: "*What are* x, y, α, $u(x)$ *and* $g(y)$? *What does the point* $\langle d_1, d_2 \rangle$ *mean? How is the* $\arg\max$ *formula used?*" The answer looks like this:

- x is HIS slicing the pie, and α is HIS bargaining power, $0 \le x \le 1$, $0 \le \alpha \le 1$;
- $u(x)$ is HIS expectation, for example $u(x) \equiv x$, of HIS x slicing the pie;
- y is HER slicing the pie, and $1-\alpha$ is HER bargaining power, $0 \le y \le 1$;
- $g(y)$ is HER expectation, for example $g(y) \equiv \sqrt{y}$, of HER y slicing the pie.

In widely accepted vocabulary, we call $s = \langle u(x), g(y) \rangle$ the utility pair. The disagreement point $d = \langle d_1, d_2 \rangle$ is what HE and SHE collect if they disagree how to slice the pie. The sugar pie disagreement point is $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$; disagreeing players collect nothing. Further, we suppose that expectations from the pie are more valuable for HER indicating HER desire $g(1/2) = \sqrt{1/2} = 0.707$, which is greater than HIS desire $u(1/2) = 0.5$.

[1] Kalai, E. (1977) Nonsymmetric Nash Solutions and Replications of 2-Person Bargaining, International Journal of Game Theory 6, 129-133.

Now considering the $\arg\max$ formula of $f(x,y,\alpha)$ one may ask a new question: "*What standard will HE, the sugar pie negotiator, base HIS decision on to obtain an equal half of the pie?*" That is to ask, what standard will facilitate HIS negotiating power α to obtain the half of the pie if SHE may only accept or reject the proposal. A technical person can shed light on the solution. First, replace $u(x)$ with x, put $y=1-x$, replace $g(y)$ with $\sqrt{1-x}$, and take the derivative of the result $f(x,1-x,\alpha)$ with respect to the variable x by evaluating $f'_x(x,1-x,\alpha)$. Later, replace $x=\frac{1}{2}$, and finally solve the equation $f'_x(\frac{1}{2},\frac{1}{2},\alpha)=0$ for α; the equation $f'_x(\frac{1}{2},\frac{1}{2},\alpha)=0$ resolves for $\alpha=1/3$.

In general, one might feel comfort in the following judgment. "*Even in the face of the fact that SHE is twice as tough a negotiator,* [2] *to count on the half of the pie is a realistic attitude towards HIS position of negotiations. Surely, rather sooner than later, since HE revealed that SHE likes sweets, HE would have HER to agree to a concession.*" This attitude might well be the standard if a half of the pie is desirable as a specific outcome of negotiations.

Exercise. In our sugar pie game, rational actors have no reason to order the pie prior to the agreement is accomplished. The negotiators must first play the bargaining drama of alternative offers, which results in some commitments of how to slice the pie. When agreement upon the slice is reached a new *real life problem* arise: Who is going to order the pie in the bakery, to decide the size, and to secure a safe delivery of the pie to its end destination. Usually the players negotiate upon matters when there are equal preconditions in place that guarantee equal privileges in the bargaining. We proceed in other direction, in a normative manner provided that, making an order of the pie, the decision-making about the size of the pie is an exclusive privilege of the first player, because we assume to deal with asymmetric information of players. However, the second player will preserve its advisory rights regarding the size of the pie.

Indeed, assume that only HE has all *relevant information of cooking*, whereas SHE does not; HE possesses some hidden skills, which can be used in situations where HE can enforce the *bakery to cook properly*, or effectively retaliate for breaches if the quality of cooking does not meet its goal whereas SHE cannot. Moral-Principal problem may also arise, because we suppose in addition that HE acts on HER behalf. Having less information about the cooking SHE cannot completely monitor

[2] Let us say, she pays her solicitor twice as much as he does.

HIS actions or intentions; HE may have an incentive to act inappropriately (from HER viewpoint) if the interests of both are not aligned in eventual agreement. We suppose that SHE lacks such abilities and knowledge and might show willingness to agree or, at least, not to resist HIS privileges to make an order and to decide upon the size of the pie. On the other hand, the bakery has limits of its own within realistic utopia, e.g., the size of the pie might be too large to fit into the oven, or deficit of finance may occur forcing the bakery to close its activities, etc., i.e., a risk of bankruptcy (fiscal inconsistency) event is pending. Bakery utopian limitations (in real life the tax system inconsistency or volatility of economic resources) are a common knowledge, as we suppose they are; both players know all these circumstances, which must be taken into account. Therefore the risk of breakdown may be the driving force for both players to order and to bake the pie reasonably.

Suppose now that in the background of HIS judgment the quality of the pie first increases when the size is small, but reaching the peak point it starts to decline: the quality, one will say, is $\cap$-single peaked upon the size. For HER the pie is desirable whatever it is. Let us try to play the sugar pie game in a different way when HE alone *placing an order for the delivery to the bakery* prescribes the size z of the pie. In the example SHE *just recommends* the size z that HE is not committed to accept, but HE is committed to the slice x aligned by the agreement. Let the utility pair $\langle u,g\rangle$ of HIS and HER expectations is given by:

$$u(\mathrm{z},x)=\mathrm{z}\cdot[(1+x/2)-\mathrm{z}]\ ,\ g(\mathrm{z},y)=\mathrm{z}\cdot\sqrt{y}\ ,\ \mathrm{z}\in[0,1],\ x,y\in[0,1].$$

Define binding slices to the size z as a curve $x(\mathrm{z})$, which resolve $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},x)=0$ for x. Evaluating x from $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},x)=0$ and then replacing $x(\mathrm{z})$ into $u(\mathrm{z},x)$ and $g(\mathrm{z},x)$ we get $u(\mathrm{z})=\mathrm{z}^2$ and $g(\mathrm{z})=\mathrm{z}\cdot\sqrt{3-4\cdot\mathrm{z}}$. Hereby, the bargaining problem $\langle S,d\rangle$ transfers into parametric space $S_{\mathrm{b}}=\langle u(\mathrm{z}),g(\mathrm{z})\rangle$ of the size parameter $\mathrm{z}\in[\frac{1}{2},\frac{3}{4}]\subset[0,1]$. I call the interval $[\frac{1}{2},\frac{3}{4}]$ by the scope of negotiations: $\frac{1}{2}$ and $\frac{3}{4}$ resolve $\mathrm{u}'_{\mathrm{z}}(\frac{1}{2},0)=0$ and $\mathrm{u}'_{\mathrm{z}}(\frac{3}{4},1)=0$ for z accordingly. In HIS view the pie must fit to size, since outside the interval $[\frac{1}{2},\frac{3}{4}]$ it has a low quality of baking but useful. Therefore, the disagreement occurs at $d=\langle u(\frac{1}{2}),g(\frac{3}{4})\rangle=\langle\frac{1}{4},0\rangle$. When we impose a constraint that the size of the welfare pie remains fixed (*stable*) during the delivery to its end destinations, the Nash symmetric solution to the game is found at $\mathrm{z}=0.69$, $x=0.74$. However, HIS asymmetric power 0.212 is sufficient to negotiate with HER about the half of the pie provided the size $\mathrm{z}=0.62$ is suitable for cooking in the oven.