Calculus of Bargaining Solution on Boolean Tables

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Abstract. This article reports not only a theoretical solution of bargaining problem as used by game theoreticians but also an adequate calculus. By adequate calculus we understand an algorithm that can lead us to the result within reasonable timetable using either the computing power of nowadays computers or widely accepted classical Hamiltonian method of function maximization with constraints. Our motive is quite difficult to meet but we hope to move in this direction in order to close the gap at least for one nontrivial situation on Boolean Tables.

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> *"Rawls second principle of justice: The welfare of the worst-off individual is to be maximized before all others, and the only way inequalities can be justified is if they improve the welfare of this worst-off individual or group. By simple extension, given that the worst-off is in his best position, the welfare of the second worst-off will be maximized, and so on. The difference principle produces a lexicographical ordering of the welfare levels of individuals from the lowest to highest."* Cit. Public Choice III, Dennis C. Mueller, p.600

1. Introduction

Since publishing in 1950 "The bargaining problem" by John F. Nash, Jr. its framework has been developed in different directions. Such was the Martin Osborn and Ariel Rubinstein "Bargaining and Markets" monograph (1990), where the Nash original "axiomatic" idea was extended to incorporate a "strategic" bargaining process as it actually happens in real life, and where the "time shortage" for the bargainers is the major factor encouraging agreements. A lot of bargaining problem varieties, decades after the Nash discovery, has been under the "loop" of many game theoreticians, where the bargaining problem solution did not necessarily comply with all Nash axioms. Beyond any doubt, "*Nonsymmetrical Solution*", Kalai (1977); "*Bargaining under Incomplete Information*", Hursanyi (1967); "*Experimental Bargaining*", Roth (1985); "*Bargaining and Coalition*", Hart (1985) etc., can extend this list to convince the reader once again in fundamental importance of bargaining theory.

Bargaining and rational choice mechanisms are related issues and such they are in our case. Along the lines of general choice theory the choice act can be formalized in two dif-

[*] Monotonic Systems idea, different from all known ideas with the same name, was initially introduced in http://www.datalaundering.com/download/modular.pdf.

ferent languages known as internal and external descriptions. Internal description uses the language of binary relations while the external explains the same properties on the set theoretical level. Both the internal and the external descriptions deal with the same object highlighting it from different angles. The Nash Bargaining Problem and its Solution express exactly the same phenomenon. Given a list of axioms, like “Pareto Efficiency” or “Independence of Irrelevant Alternatives”, in terms of binary relations, which the rational actors must follow, the solution necessarily is a scalar optimization on the set of alternatives. Exactly, the scalar optimization keeps the secret of whole Nash’s axiomatic approach and its success in performing the calculus of bargaining solution. In connection with the bargaining, as well, the motive of our paper is to report a “calculus” of bargaining solution on large Boolean tables and some theoretical foundations offered by the method. Unfortunately we met a lot of difficulties in following the Nash’s scenario.

Boolean table representation discloses the real life “cacophonous” into relatively simple and understandable data format. However, it complicates the picture allowing the scalar optimization not to be unique. Moreover, we are dealing here with an object purely atomic without any hope from the first glance to implement the “invariance under the change of scale of utilities” property in the proofs. From the researcher’s point of view the situation fails from incertitude of what type scalar criteria suits best – the Nash axiomatic approach suggest the product of utilities removing the situation incertitude once and for all from any discussion. Nevertheless, we believe that a reasonable solution under the jurisdiction of our method might come into consideration while for the game-analyst to enroll the method into the arsenal of game analysis tools will be an advantage.

In the next section we present the main example of our bargaining game. In addition, we illustrate, in the appendix, also, a different bargaining on Boolean Tables between the coalition and its moderator using some conventional characteristic functions. Certain items in the main example, like signals or misrepresentations, should not be understood as a primary topic of our discussion. These items must rather be understood as an illustration of bargaining process complexity. In the third section we try to explain our intentions in more rigorous way. Here we formulate our “Bargaining Problem on Boolean Tables” in pure strategies raising the foundation for section 4, where we are going to exploit our pure Pareto frontier in terms of so-called Monotonic Systems chain-nested alternatives – the

Frontier Theorem. In order to implement the Nash theorem for nonsymmetrical solution, Kalai 1977, in the Section 5, we introduce acceptable backbreaking algorithm in general form. Despite the lotteries are not allowed upon Boolean Tables subsets of pure strategies and which do not necessary arrange a convex collection of feasible alternatives as usual, we claim anyway that the algorithm will find an acceptable solution. At last, as promised above, the Section 6 corresponds to an elementary attempt to formulate a regular approach of coalition formation under the coalition formation supervisor – the moderator. Exactly, this attempt visualized on Figure 2, explain the notation vocabulary of chain-nested alternatives prevailed in our Monotonic Systems theory, see also Section 4. Section 7 explains the whole story alongside of independent heuristic interpretation. In few words we conclude the study, Section 8.

2. Example.

Company "Well-Being" manager is determined to encourage employees' health activities. The manger hopes to reduce company losses with regard to disability compensations. To find the employees preferences the manager recommends proceeding with a survey. The survey disclosed, that employee's promised to attend health activities in accordance with the Table 1.

Table 1

Health activities	*No Smoking*	*Swimming Pool*	*Bike Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Empl. nr.1		x	x			2
Empl. nr.2	x	x		x	x	4
Empl. nr.3		x	x	x		3
Empl. nr.4	x	x		x	x	4
Empl. nr.5			x	x		2
Empl. nr.6	x	x	x	x	x	5
Empl. nr.7		x	x			2
Total	3	6	5	5	3	22

The manager believes the employees, but s/he is aware about their unreliable human nature in keeping their promises. Therefore the manager decides to award employees, which will participate in health activities and which will be organized in "Health Club".

The manager has found a sponsor accepting 12 Bank Notes in cover for the project expenses. But, thinking over the awards policy, the manager realized that there are a lot of obstacles.

For the first, it is not very practical to organize activities with only few participants. If organizing and wasting recourses it is desirable to follow a health policy agreed to number of employees. For example, it is not a good idea to waste resources on activity with only one participating employee. On the other hand, it is desirable to elaborate acceptable list of activities encouraging many employees in keeping their promises. For the second, it is not worthwhile to enforce numerous instructions (as a rule full of twists and turns) regarding the awards regulations. Usually, in such situation, someone must be in command – a moderator, who will be responsible for the club formation. However, it seems natural for the manager not to loose control over situation totally, because the manager is also responsible for health activities financing. Thus, the manager proposes to write down the first club regulation: *the manager awards* 1 *Bank Note to an employee participating in at least* k *different activities*. Hereby the parameter k is in place for the manager choice.

The parameter k choice is a delicate matter. The manager task is not exactly about employees' preferences regarding specific activities to participate. Most accurately this task is in the moderator jurisdiction; as well of course, it is a task of employees themselves – now the club members. The delicacy of the situation is to prohibit some club members to "spring over" health activities preferred by other members of the club by worsening, in manager's view, the situation legalizing too many different activities to be materialized. Here comes the second club regulation: *if certain employee in favor of receiving awards participated in less than* k *activities no one be awarded* – all participants tried in vain to settle a club, inclusive the moderator, who's award rule is lit a bit tangled by the third regulation. To put into operation the third regulation, the manager hopes to encourage the moderator to exclude activities with small number of participants: *moderator's award basket will be equal to the lowest number of participants in the activity list of club activities materialized by club members*. Indeed, to earn more, the moderator might organize a new club by excluding an activity with the lowest number of participants from activities of some, already organized club so that the lowest number of participants in the new and shorter list is higher than in the previous list. Note also, what does not count in award regulation, that if a club member declines an activity, it may happen that someone outside the club participates in that activity. So, the club "activities list" may get shorter than in Table 1 to decide the size of moderator's award.

Worse yet. There exists an opportunity for misrepresentation of club members preferences before the company manager. Suppose that the manager makes a decision $k = 1$. This decision by mistake, or other reason, has been made accessible to the moderator. Knowing that $k = 1$ it is easy to predict the moderator actions in accordance with the third club regulation. Indeed, from the survey results the moderator might find out the most "popular" health activity and which employees exactly intend to participate in this activity. The moderator, as it is quite clear, can count on the maximum award persuading employees to become club members by an offer to participate only in that particular activity. Our rational members certainly agree on that, because "participating or not" in other activities the same award is still guaranteed [1]. Such like offers, obviously, have a chance to occur for $k > 1$ as well.

The whole story essence lies in the moderator award. If not the moderator award, the grand coalition formation is immanent – all employees will became club members. Yes of course, it should be clear that each employee could guarantee his/her own award participating at least in one or few activities. Such grand coalition formation is not always feasible due to the moderator actions. Really, as we already pointed out, the moderator might receive a minor award by the reason of someone "curious" employee still participating in that "unpopular activity". Isn't it true that, following our third club regulation, the moderator award size is set up depending exactly on that "unpopular activity"?

Being a rational actor, the company manager, from all just said, will keep the decision k in secret. On the contrary, it is reasonable to believe that all parties involved – the club members, the moderator and the manager, have their own preferences upon the number k. Therefore, an explanation in view of salon game suits well in our situation. Suppose that the manager has chosen a card k hiding it from the counter partners. Let the moderator and the club members have reached an agreement on their own card choice complying with our three club regulations. The game is over with awards payoffs in case the manager

[1] We will disclose more complex misrepresentation opportunity later.

card is not higher than the counterpart card. Otherwise none awards will be paid off despite the club formation has been taken into consideration.

But, we have not told the whole truth yet. The positive effect f_k, which the manager hopes to achieve, depends on the decision k. We have to expect a single $\cap$-peakedness of the effect function for some reason. As a result, this function separates the region of k values into what we call prohibitive and normal range. In the prohibitive range – the low values of k-numbers – the effect has not yet reached its maximum value, but f_k top level has been stepped over in the normal range – the high k-values. Therefore, in the prohibitive range the manager and the moderator interests compete each other and here it is reasonable for the manager to keep his decision in secret. However, in the normal range they might cooperate not allowing too high k-values, because both of them can loose their payoffs. So, in the normal range it is not of the manager best interests to hide the k-card.

We are now at the point where we can illustrate the game situation more closely. Let us take a look at the Table 1 and let the award will be granted at $k = 1{,}2$. The manager may count upon all 7 employees to become the members of the club to participate in all activities; each of them awards a Bank Note, the moderator's basket size is equal to 3. However, the moderator may suggest to the club members to drop *"No Smoking"* and *"Fattening Diet"* activities in order to raise his own award to 5. All club members will preserve theirs awards – sounds the moderator's argument, see the Table 2.

Table 2

Health activities	*Swimming Pool*	*Bike Exercises*	*Moderate Alcohol*	*Total*
Empl. nr.1	x	x		2
Empl. nr.2	x		x	2
Empl. nr.3	x	x	x	3
Empl. nr.4	x		x	2
Empl. nr.5		x	x	2
Empl. nr.6	x	x	x	3
Empl. nr.7	x	x		2
Total	6	5	5	16

Table 3

Swimming Pool	*Total*
x	1
x	1
x	1
x	1
	0
x	1
x	1
6	6

One can see that the total awards expenses to organize the club may now increase up to 12 Bank Notes for the sponsor. Sponsor may also insist that $k = 1$ is undesirable from an additional intersection, since the moderator can misrepresent the members' preferences [2]. The sponsor is anxious that moderator may offer one Bank Note to someone from employee's board for signaling the decision $k = 1$. Knowing that $k = 1$ the moderator may propose to the club members to drop all except *"Swimming Pool"* activities. However in sponsor opinion the moderator must compensate nr.5 employee losses by one Bank Note. If not, the employee nr.5, participating in activities distinct to *"Swimming Pool"*, has a right to receive an award and may send a signal to the board regarding moderator's fraud. Moderator's award in this case, following the regulations rules in force (see Table 3), will be 6 minus 1 for the signal, and minus 1 for the compensation: it makes 4 what is greater than the Table 1 suggests. In order to decrease sponsor expenses or to avoid misrepresentations company board may follow the sponsor's advice and propose to choose the $k \geq 3$.

One may argue that $k \geq 3$ yields fewer participation in health activities because the employees' nr. 1, 5 and 7 will be excluded from the club and will immediately drop all their activities. However, someone else may counter argue that if it happens, as anyone can see from the Table 4 below, remaining employees $2,3,4$ and 6 will still participate in heath activities and will still be awarded.

Table 4

Health activities	*No Smoking*	*Swimming Pool*	*Bike Exer-cises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Empl. nr.2	x	x		x	x	4
Empl. nr.3		x	x	x		3
Empl. nr.4	x	x		x	x	4
Empl. nr.6	x	x	x	x	x	5
Total	3	4	2	4	3	16

Now the moderator's award basket equals 2, since the employees' nr. $3,6$ alone prefer to participate in *"Bike Exercises"*. After all, the sponsor expenses decrease from 10 to 6.

[2] Here comes the more complicated misrepresentation as promised.

But now, the manager may compromise with the moderator in keeping his award still equal to 3 while dropping *“Bike Exercises”* since organizing an activity attended only by two participants might be worthless anyway, see Table 5. Note that employee nr. 3 following the compromise must be excluded from the club list by the second club regulation, c.f. the suggestion above to drop *“No Smoking”* and *“Fattening Diet”* activities.

Table 5

Health activities	*No Smoking*	*Swimming Pool*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Empl. nr.2	x	x	x	x	4
Empl. nr.4	x	x	x	x	4
Empl. nr.6	x	x	x	x	4
Total	3	3	3	3	12

Is the decision reasonable? Suppose not. Let $k = 5$ will be the board proposal. Now, only the employee nr.6 is ready to participate in health activities, see Table 6.

Table 6

Health activities	*No Smoking*	*Swimming Pool*	*Bike Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Empl. nr.6	x	x	x	x	x	5
Total	1	1	1	1	1	5

The moderator may disagree to organize the club, because his award is only one Bank Note. A posteriori, it is not exactly the manager’s stimulus either to promote all 5 activities with only one participant. As a result, the manager votes against the proposal $k = 5$ at the board meeting. To reach a conclusion, the basic nature of manager’s difficulty consists of how to choose an alternative k from the list below, see Table 7.

Table 7.

	Club members	*Club moderator*	*Club members compensation*	*Signal*	*Bank Notes used*	*Bank Notes left*
Tab. 1, $k = 2$	7	3	0	0	10	2
Tab. 2, $k = 2$	7	5	0	0	12	0
Tab. 3, $k = 1$	6	4	1	1	12	0
Tab. 4, $k = 4$	3	1	0	0	4	8
Tab. 5, $k = 4$	3	3	0	0	6	6
Tab. 6, $k = 5$	1	1	0	0	2	10

To make these things crystal clear, we visualize the manager difficulty by a bargaining game to share 12 Bank Notes between the moderator and the club members.

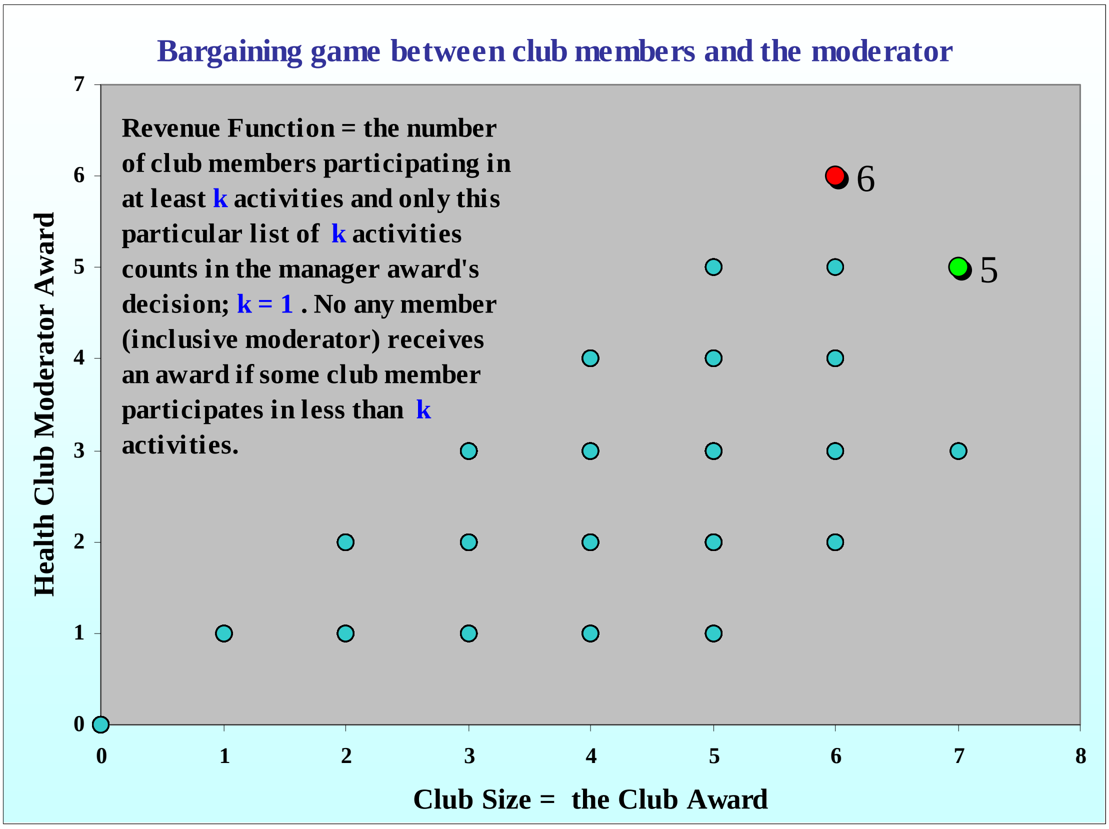

Figure 1.

Our section ends here without telling the whole truth what was the decision k at the board meeting. We will tell the truth in rigorous vocabulary in next sections. Only a closing topic is necessary to interrupt our pleasant story for a moment [3].

Let our three actors have been engaged in interaction: employees N, moderator in charge of club formation and the manager. Certain employees from $N = \{1,...,i,...n\}$ – the coming members of the club x, $x \in 2^N$, have expressed their willingness to participate in certain activities y, $y \in 2^M$, $M = \{1,..., j,...,m\}$. Let a Boolean table $W = \| a_{ij} \|_n^m$ reflects the survey result of employees' preferences; $a_{ij} = 1$ if employee i has promised to participate in activity j, $a_{ij} = 0$ if not. Also 2^M lists of allegedly subsidized activities $y \in 2^M$ have been examined.

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[3] For those unwilling to continue with bargaining in next sections please pay attention to this closing remark.

We can calculate the moderator payoff $F_k(H)$ using a subtable H on crossing entries of the rows x and columns y in the original table W by further selection of a column with the least number $F_k(H)$ from the list y. The number of 1-entries in each column belonging to y determines the payoff $F_k(H)$. Characteristic functions family $\nu^k(x,y) \equiv \nu^k(H)$, $k \in \{1,...,k,...,k_{max}\}$, on N are known for the coalition games, in particular for every pair $L \subset G$, $L,G \in 2^N \times 2^M$, we suppose that $\nu^k(L) \le \nu^k(G)$. One might find it not difficult to imagine that the manager payoff function $f_k(H)$ has a single $\cap$-peakedness shape within the line of decisions $\langle 1,...,k,...,k_{max} \rangle$; $f_k(H)$ reflects some kind of positive effect on the company deeds. Sponsor expenses will be equal to $\nu^k(H) + f_k(H)$.

Finally, we share some ideas for reasonable solution of our game. The situation is similar to the Nash Bargaining Problem from 1950, where two partners – the club members and the moderator try to find a fair agreement. It is possible to find the Bargaining solution $S_k \in \{H\} = 2^N \times 2^M$ for each particular decision k, see next sections. However, the choice of the number k is something different. We have pointed out in the example that the choice $k = 4,5$ may be useful from some ex-ante reasoning. Maximum payoffs are guaranteed for the partners when the choice $k = 1$. Counting on that decision is irrational, because here only one activity will be materialized with the maximum number of participants, but without a positive effect $f(S_k)$ on the health deeds in general. The choice of higher k is either counterproductive – a lot of different activities will be at hand, but with lower number of participants, what is useful only for the sponsor in saving awards funds. For example, for $k = k_{max}$ an employee with the largest list of preferred k_{max} activities might become the only member of the club. As it seems to us the situation here is like a median voter scheme, see Barbera et. al., 1993. However, a consultation in this “white field” is necessary.

3. Bargaining game on Boolean Tables

Suppose that employees who intend to participate in company activities have been interviewed in order to reveal their preferences. The resulting data can then be arranged in $n \times m$ table $W = \|\alpha_{ij}\|$, where the entry $\alpha_{ij} = 1$ if an employee i promises to participate in activity j, $\alpha_{ij} = 0$, if not. In this respect, the primary table W is a collection of Boolean columns, where each column is filled with Boolean elements from only one particular activity. In the context of the bargaining game, we discuss an interaction between the health club and the moderator. The club choice x is a subset of rows $\langle 1,...,i,...,n \rangle$ - the coming members of the club, and a subset y of columns $\langle 1,...,j,...m \rangle$ is the moderator's choice – the coming list of activities. The result of interaction is a subtable H or a block. This block represents the players' joint anticipation (x,y). The players: Nr.1 – the club, and player nr.2 – the moderator, both of them, have in mind to receive the awards. Employees approve our three awards regulation [4]. Despite that both players are interested in company activities their objectives are different. The player nr.1 objective might be for each member of the club a higher participation in coming activities. The player nr.2, the moderator's objective, might be the higher participants' number attending each activity arranged by the company. Let the utility pair $(v(x),F(y))$ highlights the players' payoff. Players have in mind to bargain upon all possible anticipations (v,F).

Our intention in developing a theoretical ground for our story is to follow the Nash's (1950) axiomatic approach. Unfortunately, as it was noticed before, there are some fundamental difficulties in similar approach. Below we summarize these difficulties step by step putting forward an equivalent. We will consequently succeed in this direction where we first formulate the Nash's axioms in their original vocabulary and then reexamine their essence in our own vocabulary. Such advance along this way will be easier in raising the fundament of proofs in next sections.

It is not a secret for anyone following Nash that "... we may define a two-person anticipation as a combination of two one-person anticipation. ... A probability combination of two two-person anticipations is defined by making the corresponding combinations for

[4] We recall the main regulation that none club members, inclusive the moderator, receive their awards if certain club member participates in less than k activities.

their components," see Nash 1950, p. 157, Sen Axiom 8*1, p. 127 or sets of axioms, see also Luce and Raifa 1958, p.25, G. Owen 1968, section VII.2 , or von Neumann and Morgenstern 1947, utility index interpretation. Rigorously speaking the compactness and convexity of a feasible set $\boldsymbol{S}$ of utility pairs ensures that any continuous and strictly convex function on $\boldsymbol{S}$ reaches its maximum while convexity guarantees the maximum point uniqueness.

Let us recall the other Nash axioms. The solution must comply with INV) invariance under the change of scale of utilities; IIA) independence of the irrelevant alternatives; and PAR) - Pareto efficiency. Note that following PAR the players object an outcome s when there is available an outcome s' in which both of them are better off. We expect for players to act from a *strong individual rationality* principle SIR. An arbitrary set $\boldsymbol{S}$ of the utility pairs $s = (s_1, s_2)$ can be the outcome of the game. A disagreement event occurs at the point $d = (d_1, d_2)$ where both of players obtain the lowest utility they count on – the status quo point. A *bargaining problem* is a pair $\langle \boldsymbol{S}, d \rangle$[5] and there exists $s \in \boldsymbol{S}$ such that $s_i > d_i$ for $i = 1,2$ and $d \in \boldsymbol{S}$. A *bargaining solution* is a function $f(\boldsymbol{S}, d)$ that assigns to every bargaining problem $\langle \boldsymbol{S}, d \rangle$ a unique element of $\boldsymbol{S}$. The bargaining solution f satisfies SIR if $f(\boldsymbol{S}, d) > 0$ for every bargaining problem $\langle \boldsymbol{S}, d \rangle$.

Our secret, which guarantees the same properties, lies in the following. We define a feasible set $\boldsymbol{S}$ of anticipations, or in more convenient vocabulary, a feasible set $\boldsymbol{S}$ of alternatives as a collection of table W blocks: $\boldsymbol{S} \subseteq 2^W$. Similar to disagreement event in Nash scheme we define an empty block $\varnothing$ to be a status quo option in any set of alternatives $\boldsymbol{S}$, which we call the refusal of choice. Given any two alternatives H and H' in $\boldsymbol{S}$ an alternative $H \cup H'$ does belong to $\boldsymbol{S}$. In other words the set $\boldsymbol{S}$ of feasible alternatives in our case always arrange an upper semilattice. Moreover, if an alternative $H \in \boldsymbol{S}$ then its all subsets $2^H \subseteq \boldsymbol{S}$. Although a room for discussion is at hand, we state that this is our equivalent to the convex property and will play the same role in proofs as it does in Nash scheme.

5 We use the bold notifications $\boldsymbol{S}$ close to the originals. Notification S is preserved for stable point, see later.

The Nash theorem asserts that there is a unique bargaining solution $f(\boldsymbol{S},d)$ for every bargaining problem $\langle \boldsymbol{S},d \rangle$, which maximizes the product of the players' gains in the set $\boldsymbol{S}$ of utility pairs $(s_1,s_2) \in \boldsymbol{S}$ over the disagreement outcome $d = (d_1,d_2)$. This is so called symmetric bargaining solution which satisfies INV, IIA, PAR, and SYM – players symmetric identify, iff

$$f(\boldsymbol{S},d) = arg\,max_{(d_1,d_2)\le(s_1,s_2)}(s_1 - d_1)\cdot(s_2 - d_2). \quad (1)$$

It is difficult to say ad hoc what properties can guarantee the uniqueness of similar solution on Boolean Tables. Nevertheless, in the next section we claim that our bargaining problem on $\boldsymbol{S} \subseteq 2^W$ has the same symmetric or nonsymmetrical shape:

$$f(\boldsymbol{S},\varnothing) \equiv f(\boldsymbol{S}) = arg\,max_{H\in\boldsymbol{S}}\, v(H)^{\theta} F(H)^{1-\theta} \quad (2)$$

for some $0 \le \theta \le 1$ provided that Nash axioms hold.

4. Theoretical aspects of the Boolean game

Henceforth, the table $W = \|\alpha_{ij}\|$ will be the Boolean table, see above, representing employees' promises to attend company activities. We suggest looking at H rows x, symbolizing the coming members of the club participating in at least k activities. Activities arrange, what we call here, a column's activity list y, $k = 2,3,...$; k is the award decision. For each activity in the activity list y at least $F(H)$ of club members intend to fulfill their promises. Let, for example, the number of rows in H is the gain $v(H)$ of player nr.1 – the club members –, while the gain of the player nr.2 – the moderator's award – is $F(H)$.

Let us look at the bargaining problem in conjunction with players' preferences. The anticipations of the coming club members $i \in x$ towards the activity list y can easily be "raised" by $r_i = \sum_{j\in y}\alpha_{ij}$ if $r_i \ge k$, and $r_i = 0$ if $\sum_{j\in y}\alpha_{ij} < k$, $i \in x$, $j \in y$. Similarly, the moderator's anticipation towards the same activity list y, can be "accumulated" by means of table H as $c_j = \sum_{i\in x}\alpha_{ij}$, $j \in y$.

We now consider the whole story in more rigorous mathematical form. Below we use the notation $H \subseteq W$. The notation H contained in W will be understood in an ordinary set-theoretical vocabulary, where the Boolean table W is a set of its Boolean 1-elements. All 0-elements will be dismissed from the consideration. Thus, H as a binary relation is also a subset of W. Below, referring an element, we assume that it is a Boolean 1-element.

For an element $\alpha \equiv \alpha_{ij} \in W$ in the row i and column j we use the similarity index $\pi_{ij} = c_j$, counting only on Boolean elements belonging to H, $i \in x$ and $j \in y$. The value of $\pi_{ij} = c_j$ depends on each subset $H \subseteq W$ and we may therefore write $\pi_{ij} \equiv \pi \equiv \pi(\alpha, H)$; the set H is called the π-function parameter. Our similarity indices π_{ij}, as one can see, may only concurrently increase with the "expansion" and decrease with the "shrinking" of the parameter H. This leads us to the fundamental definition.

Definition 1. Basic monotone property. *By a Monotonic System will be understood a family $\{\pi(\alpha, H) : H \in 2^W\}$ of π-functions, such that the set H is to be considered as a parameter with the following monotone property: for two particular values $L, G \in 2^W$, $L \subset G$ of the parameter H the inequality $\pi(\alpha, L) \le \pi(\alpha, G)$ holds on all elements $\alpha \in W$.* In ordinary vocabulary the *π-function* with the definition area $W \times 2^W$ is monotone on W with regard to the second parameter on 2^W.

Definition 2. Let $V(H)$ for a non-empty subset $H \subseteq W$ by means of a given arbitrary threshold u is the subset $V(H) = \{\alpha \in W : \pi(\alpha, H) \ge u\}$. *The non-empty H-set indicated by S is called a stable point with reference to the threshold u if $S = V(S)$ and there exists an element $\xi \in S$, where $\pi(\xi, S) = u$.* See Mullat (1979,1981) for a comparable concept. Stable point $S = V(S)$ has some important properties, which cannot be left apart, see later.

Definition 3. *By Monotonic System kernel we understand a stable point $S^* = S_{max}$ with the maximum possible threshold value $u^* = u_{max}$.*

Similar properties of Monotonic Systems and their kernels are under investigation, see Libkin et al. (1990), Genkin et al. (1993), Kempner et al. (1997), Mirkin et al. (2002). With regard to current investigation we have to make what we believe an important comment. Given a Monotonic System in general form, without any reference to any kind of "interpretation mechanism", one can always consider a bargaining game between a coalition H – the player nr.1, with characteristic function $v(H)$, and the player nr.2 with the payoff function $F(H) = min_{\alpha \in H}\, \pi(\alpha, H)$. Following Nash theorem, a symmetrical solution has to be found in form (1). Below we are going to prove as well that our solution has to be found in the symmetrical or nonsymmetrical form (2).

Definition 4. Let d is the number of Boolean 1's in table W. *An ordered sequence $\overline{\alpha} = \langle \alpha_0, \alpha_1, ..., \alpha_{d-1} \rangle$ of distinct elements in the table W is called a defining sequence if there exists a sequence of sets $W = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ such that:*

A. Let the set $\mathrm{H}_k = \{\alpha_k, \alpha_{k+1}, ..., \alpha_{d-1}\}$. *The value $\pi(\alpha_k, H_k)$ of an arbitrary element $\alpha_k \in \Gamma_j$, but $\alpha_k \notin \Gamma_{j+1}$ is strictly less than $F(\Gamma_{j+1})$, $j = 0,1,..., p-1$.*

B. *There does not exist in the set Γ_p a proper subset L, which satisfies the strict inequality $F(\Gamma_p) < F(L)$.*

Definition 5. *A defining sequence is complete, if for any two sets Γ_j and Γ_{j+1} it is impossible to find Γ' such that $\Gamma_j \supset \Gamma' \supset \Gamma_{j+1}$ while $F(\Gamma_j) < F(\Gamma') < F(\Gamma_{j+1})$, $j = 0,1,..., p-1$.*

It has been established that in arbitrary Monotonic System one can always find a complete defining sequence, see Mullat (1971,1976). Moreover, each set Γ_j is the largest stable set with reference to the threshold $F(\Gamma_j)$. Now we can formulate our Frontier Theorem.

Frontier theorem. *Given a bargaining game on Boolean tables with an arbitrary set $\boldsymbol{S}$ of feasible alternatives $H \in \boldsymbol{S}$ the anticipations points $(v(\Gamma_j), F(\Gamma_j))$, $j = 0,1,..., p$, of a complete defining sequence $\overline{\alpha}$ arrange a Pareto frontier in $\Re^2$.*

Proof. Let $W^S \in \boldsymbol{S}$ is the largest set in $\boldsymbol{S}$ containing all other sets $H \in \boldsymbol{S}:\ H \subseteq W^S$. Let a complete defining sequence $\overline{\alpha}$ [6] has been found for W^S. Let the set H^c is the set containing all such sets $V(H)$, where $V(H) = \{\alpha \in W : \pi(\alpha, H) \geq F(H)\}$. Notice that $H \subseteq V(H^c)$ and $F(H^c) \geq F(H)$. Now, to be accurate, we must distinguish between three situations: a) in the sequence $\overline{\alpha}$ one can find an index j such that $F(\Gamma_j) \leq F(H^c) < F(\Gamma_{j+1})$ $j = 0,1,\ldots,p-1$, b) $F(H^c) < F(W) = F(\Gamma_0)$ and c) $F(H) > F(\Gamma_p)$. The case c) is impossible because on the set Γ_p the function $F(H)$ reaches its global maximum. In case of b) the anticipation $(v(\Gamma_0), F(\Gamma_0))$, $\Gamma_0 = W$, is better off than $(v(H), F(H))$ what concludes the proof. In case of a) let $F(\Gamma_j) < F(H^c)$ - otherwise the equality $F(\Gamma_j) = F(H^c)$ is the statement of the theorem (read the sentence after the next and change index $j+1$ to j). But now the set H^c must coincide with Γ_{j+1}, $j = 0,1,\ldots,p-1$, otherwise the defining sequence $\overline{\alpha}$ is not complete. Indeed, looking at the first element $\alpha_k \in H^c$ in the sequence $\overline{\alpha}$ one can establish that if not $\Gamma_{j+1} = H^c$ then the set $H_k = H^c$ because it is the largest set stable up to the threshold $F(H^c)$. Hence the set H_k represents an additional Γ-set in the sequence $\overline{\alpha}$ with the property A of a complete defining sequence. Due to $\Gamma_{j+1} = H^c \supseteq H$ and the basic monotonic property the following inequalities $F(\Gamma_{j+1}) = F(H^c) \geq F(H)$ and $v(\Gamma_{j+1}) = v(H^c) \geq v(H)$ are true. Thus, the point $(v(\Gamma_{j+1}), F(\Gamma_{j+1}))$ is better off than $(v(H), F(H))$. ■

5. Calculus of the Bargaining Solution.

To summarize, we are under the jurisdiction of Nash bargaining scheme. Some reservations see, for example, Luce and Raifa, 6.6, hold as usual because our bargaining game on Boolean tables is purely atomic not allowing lotteries. Lottery is an important element of the whole story of bargaining. The bad thing is that if the lottery is not allowed, no one from the first glance can guarantee the uniqueness of the Nash solution. However, the good thing is that "...the Nash solution of $\langle \boldsymbol{S}, d \rangle$ depends only on disagreement point d and the Pareto frontier of $\boldsymbol{S}$. The compactness and convexity of $\boldsymbol{S}$ are important only

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[6] We are not going to use any new notifications to distinguish in between Boolean tables W and W^S.

insofar as they ensure that the Pareto frontier of $\boldsymbol{S}$ is well defined and concave. Rather than starting with the set $\boldsymbol{S}$, we could have imposed our axioms on a problem defined by a non-increasing concave function (and disagreement point d)...", see Osborn and Rubinstein (1990), p. 24 – in our case $\left(v(\Gamma_j), F(\Gamma_j)\right)$, $j = 0,1,...,p$, represents the atomic Pareto frontier. Therefore nothing can prevent us to implement the proof of non-symmetrical solution, see Kalai, 1977, p. 132, and to perform the calculus with the product of utility gains in its asymmetrical form (2).[7] The problem how to maximize the product is more technical one. From now on in the following we introduce an algorithm for that purpose. We will first comment the algorithm in lines with the definitions.

The algorithm's very last pass, see below, through the step **T** detects the largest kernel $\overline{K} = S^{*}$[8], Mullat 1995. The original version (Mullat, 1971) of the algorithm to detect the largest kernel looks like a greedy inverse serialization procedure, see Edmonts, 1971. Original version of the algorithm produces a complete defining sequence, what is absolutely imperative for finding the bargaining solution subordinating with the Frontier Theorem. In view of current version it produces not complete defining sequence. As one can notice, it detects only some thresholds u_j, and only some stable set $\Gamma_j = S_j$. The sequence $u_0, u_1,...$ is monotonically increasing: $u_0 < u_1 < ...$ while the sequence $\Gamma_0, \Gamma_1,...$ is monotonically shrinking: $\Gamma_0 \supset \Gamma_1 \supset ...$, the set $\Gamma_0 = W$ is stable towards the threshold $u_0 = F(W) = \min_{(i,j)\in W} \pi_{ij}$. Therefore, the original algorithm always has a higher complexity. However, for finding the bargaining solution we still can implement the lower complexity algorithm. On this purpose we need to switch the indices $\pi_{ij} = c_j$ to somewhat different.

Let us consider the problem of how to find the players joint choice H_{max} representing a block $\arg\max_{H\in\boldsymbol{S}} v(H)^{\theta} F(H)^{1-\theta}$ of the rows x and columns y in the original table W with the property that $\sum_{j\in y} \alpha_{ij} \geq k$, $i \in x$.

Let an index $\pi_{ij} = r_i \cdot v^{\theta} \cdot c_j^{1-\theta}$[9]. Following algorithm solves the problem.

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[7] There are a lot of techniques to guarantee the uniqueness of the product of utility gains. We are not going to discuss this matter at all, because this case is rather an exemption than a rule.

[8] It may happen that some smaller kernels exist as well.

[9] This index obeys the basic monotone property as well.

Algorithm.

Step I. To set the initial values.

1i. Assign the table parameter H to be identical with W, $H \Leftarrow W$. Set minimum and maximum bounds a, b on threshold u for $\pi_{ij} \in H$ values.

Step A. To find that the next step **B** produces a non-empty subtable H. Remember the current status of table H by temporary table $H°$: $H° \Leftarrow H$.

1a. Test u as $(a+b)/2$ using step **B**. If it succeeds replace a by u. If it fails replace b by u and H by $H°$: $H \Leftarrow H°$ - “regret action”.

2a. Go to **1a**.

Step B. To test whether the minimum of $\pi_{ij} \in H$ over i, j can be at least u.

1b. Delete all rows in H where $r_i = 0$. This step **B** fails if all rows in H must be deleted; proceed to **2b**. The table H is shrinking.

2b. Delete all elements in columns where $\pi_{ij} \le u$. This step **B** fails if all columns in H must be deleted; proceed to **3b**. The table H is shrinking.

3b. Perform step **T** if none deleted in **1b** and **2b**; otherwise go to **1b**.

Step T. To test that the global maximum is found. Table H has halted its shrinking.

1t. Among numbers $\pi_{ij} \in H$ find the minimum $min \leftarrow \pi_{ij}$. Test performing step **B** with new value $u = min$. If it succeeds put $a = min$, return to step **A**. If it fails, final stop.

6. Boolean game cooperative aspects

A cooperative game is a pair (N, v), where N symbolize a set of players and v is the game characteristic function. Function v is called a supermodular if

$$v(L) + v(G) \le v(L \cup G) + v(L \cap G)$$

and submodular for the inequality sign $\le$ changed to $\ge$, $L, G \in 2^N$. Among others, see also Cherenin et al. (1948), Shapley (1971), specifies various properties of supermodular set functions. In the appendix we illustrate a game, which is neither supermodular nor submodular, but somewhat mixture like game, where single and pair vise players do not receive extra awards. On the other hand, it is obvious that all properties of supermodular functions v remain untouched for submodular $-v$ characteristic function or visa versa.

A marginal contribution into coalition H of a player x (the player marginal utility) is given by $\pi(x;H) \equiv \frac{\partial H}{\partial x}$, where

$$\frac{\partial H}{\partial x} = v(H \cup x) - v(H) \quad \text{if } x \notin H \text{, the player } x \text{ joins the coalition, and}$$

$$\frac{\partial H}{\partial x} = v(H) - v(H \setminus x) \quad \text{if } x \in H \text{, the player } x \text{ leaves the coalition,}$$

for every $H \in 2^W$.

Suppose that player's x interest to join the coalition equals the player marginal contribution $\frac{\partial H}{\partial x}$. A coalition game is convex (concave) if for any pair L and G of coalitions $L \subseteq G \subseteq W$ the inequality $\frac{\partial L}{\partial x} \leq \frac{\partial G}{\partial x}$ $\left(\frac{\partial L}{\partial x} \geq \frac{\partial G}{\partial x}\right)$ holds for each player $x \in W$.

Theorem. *For the coalition game to be convex (concave) it is necessary and sufficient for its characteristic function to be a supermodular (submodular) set function.*

Extrapolated from Nemhauser, et. al. (1978). [10]

Now, in view of the theorem, marginal utilities of players in the supermodular game motivate them sometimes to form coalitions. In modular game, where the characteristic function is both supermodular and submodular, marginal utilities are indifferent to collective rationality; because of entering a coalition nobody wins or loose a side payments. On the contrary, collective rationality sometimes is counterproductive in submodular games. Therefore in supermodular games formation of too numerous coalitions might be immanent, for example the grand coalition; in Shapley's (1971) words "snowballing" or "band-wagon" effect take place. On the contrary, submodular games are less cooperative. For the reason to counteract these "bad motives" of players both in supermodular and submodular games, we introduce below a second actor – the moderator. So, we consider a bargaining game between the coalition and the moderator.

Convex (concave) game induces an accompanied bargaining game with utility pair $(v(H),F(H))$, where $F(H) = min_{x \in H} \frac{\partial H}{\partial x}$ $\left(F(H) = max_{x \in H} \frac{\partial H}{\partial x}\right)$. Coalition itself acts in player nr.1 role with the characteristic function $v(H)$. The coalition moderator – the player nr.2 award equals $F(H)$.

[10] Shapley (1971) noticed this condition as equivalent, Nemhauser, et al. (1978) have proposed similar derivatives in their investigation of some optimization problems, Muchnik and Shvartser (1987) have pointed to the link between a submodular set functions and the Monotonic Systems, see Mullat (1971).

Proposition. *The solution* $f(\boldsymbol{S},\varnothing)$ *of a Nash's Bargaining Problem* $\langle \boldsymbol{S},\varnothing \rangle$, *which accompanies a convex (concave) coalition game with characteristic function* v, *lies on its Pareto frontier* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *maximizing (minimizing) the product* $v(\Gamma_j)^{\theta} \cdot \dfrac{\partial \Gamma_j}{\partial \alpha}^{1-\theta}$ *for some* $j = 0,1,\ldots,p$, *and* $0 \le \theta \le 1$.

Proof: Statement is an obvious corollary from the Frontier Theorem. ■

In accordance with the basic monotonic property, see above, given some monotonic function $\pi(x;H) \equiv \dfrac{\partial H}{\partial x}$ on $N \times 2^N$ it is not immediately apparent that there exists some characteristic function $v(H)$ for which the function $\pi(x;H)$ constitutes a monotonic marginal utility $\dfrac{\partial H}{\partial x}$. Following theorem, accommodated in lines of Muchnik and Shvartser, answers the question.

The existence theorem. *For the function* $\pi(x,H)$, *to represent a monotonic marginal utility* $\dfrac{\partial H}{\partial x}$ *of some supermodular (submodular) function* $v(H)$ *it is necessary and sufficient that:*

$$\frac{\partial}{\partial y}\frac{\partial H}{\partial x} \equiv \pi(x;H) - \pi(x;H \setminus y) = \pi(y;H) - \pi(y;H \setminus x) \equiv \frac{\partial}{\partial x}\frac{\partial H}{\partial y}$$

holds for $x,y \in H \subseteq N$. The interpretation of this condition we leave to the reader.

7. Heuristic interpretation

Only one, the last issue, is in place regarding our bargaining solution $\Gamma = f(\boldsymbol{S},\varnothing)$ in accompanied supermodular bargaining game. The coalition Γ is a stable point with reference to the threshold value $u = F(\Gamma) = min_{x \in K} \dfrac{\partial \Gamma}{\partial x}$. This coalition guarantees a gain $u = F(\Gamma)$ to player nr.2. Therefore, by all means available to player nr.2, anyone $x \notin \Gamma$ outside the coalition $\Gamma \in \boldsymbol{S}$ will be prevented to become a new member of the coalition just because outsider's marginal contribution $\dfrac{\partial \Gamma}{\partial x}$ brings down player nr.2 guaranteed gain. The same incentive of player nr.2 will prevent some members $x \in \Gamma$ to leave the coalition. Following unconventional interpretation might highlight the situation.

In short, observe a family of functions on $N \times 2^N$ monotonic towards the second set variable H, $H \in 2^N$. Let it be a function $\pi(x;H) \equiv \frac{\partial H}{\partial x}$. We already cited Shapley, who introduced (1971) the convex games. Convex games marginal utility $\frac{\partial H}{\partial x} = v(H) - v(H \setminus x)$ is the one of many exact utilizations of suchlike monotonicity $\pi(x,L) \leq \pi(x,G)$ for $x \in L \subseteq G$. Some studies, including current research, call such like marginal $v(H) - v(H \setminus x)$ set functions the derivatives of supermodular functions $v(H)$. Inverting the inequalities we get submodular set functions.

Convex coalition game, we stress Shapley's words (1971) once again, have some kind "snowballing" or "band-wagon" effect of cooperative rationality, i.e., in supermodular game the cooperative rationality suppresses the individual rationality. In submodular games with the inverse property $\pi(x,L) \geq \pi(x,G)$, on the contrary (an extrapolation this time), the individual rationality suppresses the collective rationality. So, in both cases it is a bad thing. The good thing what may happen, see above, a moderator might be in charge for coalition formation while the moderator award will be equal to the least marginal utility $u = F(H) = min_{x \in H} \frac{\partial H}{\partial x}$ of some weakest player in the coalition H under formation. Now a two-person's cooperative drama to be performed between the moderator and the coalition.

We already approach our heuristic interpretation. Following the apparatus of monotonic systems in terms of data mining, Mullat (1971), it is reasonable to find the Pareto frontier also in terms of game theory. The moderator bargaining strategy might be. First, in the grand coalition $N \equiv \Gamma_0$, the moderator finds out the players with the least marginal utility $u_0 = F(N) = min_{x \in N} \frac{\partial N}{\partial x}$, all together. Then the moderator will tell them to stay in line and wait for their awards. All players in line will be abandoned for a moment from any coalition formation. Following the game convexity, someone new player from the remaining players (from players still remaining in the coalition formation process) must find themselves worse off owing their position to turn for the worse upon the abandoned players in line. Moderator suggests the new bad players, as well, to join the line and wait for their awards. The moderator continues the line construction. A moment 1 comes when all remaining players Γ_1 (outside the line) are better off than u_0, i.e., better off than those staying in line and who are still waiting for their awards. Now the moderator repeats the whole procedure upon players $\Gamma_1, \Gamma_2, \ldots$until all players from N stay in line being ready to get their awards. Moderator, certainly, keeps some account about the events $0,1,\ldots$ when the marginal utility thresholds jump from u_0 to u_1, etc., occurs. It is obvious that the jumps occur only upwards: $u_0 < u_1 < \ldots < u_p$.

What happens? Players staying in line arrange a nested sequence of coalitions $\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \rangle$. Most powerful marginal players, the players when the very last event p happens, form a coalition Γ_p. The next powerful coalition will be Γ_{p-1}, etc., coming back once again to the start event 0, when the players arrange the grand coalition $N = \Gamma_0$. Our Frontier Theorem guarantees that suchlike moderator bargaining strategy, in convex games, classifies a Pareto frontier $\langle (\nu(\Gamma_0), u_0), (\nu(\Gamma_1), u_1), \ldots, (\nu(\Gamma_p), u_p) \rangle$ for bargaining game between the moderator and coalitions under formation. [11] So, the game ends with bargaining agreement between the moderator and the coalition. However, some bad players might still stay in vain waiting for their awards, because the moderator might not agree to allow them playing a role in coalition formation. Yes indeed, just on those marginal players account the moderator may loose a lot of his award $F(\Gamma_k)$, for some k's $\in \langle 1, \ldots, p \rangle$. [12]

8. Conclusion.

Nash bargaining solution being understood as a point on the Pareto frontier in Monotonic System might be an acceptable convention in the framework of "fast" calculation. The corresponding algorithm for finding the solution is characterized by a relative few operations and by known computer programming "recursive techniques" on tables. From theoretical point of view we believe that our technique represent an object to be noticed in the laboratory of game theoreticians. However, our bargaining solution is not yet totally built on already validated scientific facts established in game theory. Consultations with the specialists of the field are necessary. We feel that our coalition formation games, one way or the other, become sufficiently clear, and do not need specific economic interpretations. Nevertheless, they need to be confirmed by other fundamental studies.

[11] This sequence of players/elements in line arranges so-called defining sequence in data mining process.

[12] We refer to such players in "Bargaining Game Fiction about Welfare State, Poverty Line and Taxpayers", see http://www.datalaundering.com/download/txdesign.pdf, as agents registered under the social security administration.

Appendix. Club formation bargaining game with neither supermodular nor submodular characteristic function. Illustration

Recall the health club formation game from section 2. Given the characteristic function $v(H)$ this time it will be secondary for us whether the club members actually arrive at individual payoffs or not, but the club formation is still of our interest. Let the game participants $N=\{1,2,3,4,5,6,7\}$ try to organize a club. Let the characteristic (revenue) function comply with the promises of employees to participate in health activities in accordance with the survey, see table 1, however we demand that all 5 health activities be materialized.

Define $v(H)=|H|+\sum_{x\in H}\sum_{j=1}^{5}a_{x\,j}$, where $H\subseteq N=\{1,2,3,4,5,6,7\}$.

In other words, a promise fulfilled by the club member contributes a Bank Note to the player. In addition to all the promises fulfilled a side payment per capita is available. By this rule $v(\{1\})=3$, $v(\{2\})=5,\ldots$ Nonetheless, we are going to change the side payments rule at once so that the game turns into neither supermodular nor submodular game. Notice, that $\sum_{i}^{7}v(\{i\})=v(N)=v(\{1,2,3,4,5,6,7\})=29$, what yields the game to be not essential. Yes, indeed, the employees, cooperating or not, will be discouraged to form a club arriving at the same gains. To change the situation similar to "*the real life cacophonous*", let the side payment per capita be removed for single and pair vise players keeping the awards in tact for all other coalitions, which size is grater than 2. Thus $v(\{1\})=2$, $v(\{2\})=4$, $v(\{1,2\})=6$, $v(\{3,6\})=5$, $v(\{2,3,5\})=12$, etc… Moderator's gain, which was defined as $F(H)=\min_{x\in H}\frac{\partial H}{\partial x}\equiv(v(H)-v(H\setminus x))$, see above, makes the employees "cooperative behavior" close to grand coalition less profitable for moderator. Therefore, we hope that the moderator will encourage employees to enter the club with "reasonable size". Examine such like phenomenon observing some values of moderator gains $F(H)$ in the table below.

Table 8.

Health Clubs List							Marginal Utilities p/capita							x	y
1	2	3	4	5	6	7	1	2	3	4	5	6	7	$v(H)$	$F(H)$
*							2							2	2
	*							4						4	4
*	*						2	4						6	2
		*							3					3	3
*		*					2		3					5	2
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
		*		*					3		2			5	2
*		*		*			5		6		5			10	5
	*	*		*				7	6		5			12	5
*	*	*		*			3	5	4		3			15	3
			*	*						4	2			6	2
*			*	*			5			7	5			11	5
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	.	*	*	*	*	*		.	4	5	3	6	3	21	3
*	.	*	*	*	*	*	3	.	4	5	3	6	3	24	3
.	*	*	*	*	*	*	.	5	4	5	3	6	3	26	3
*	*	*	*	*	*	*	3	5	4	5	3	6	3	29	3

At last, we illustrate the bargaining game by the graph below and make some comments.

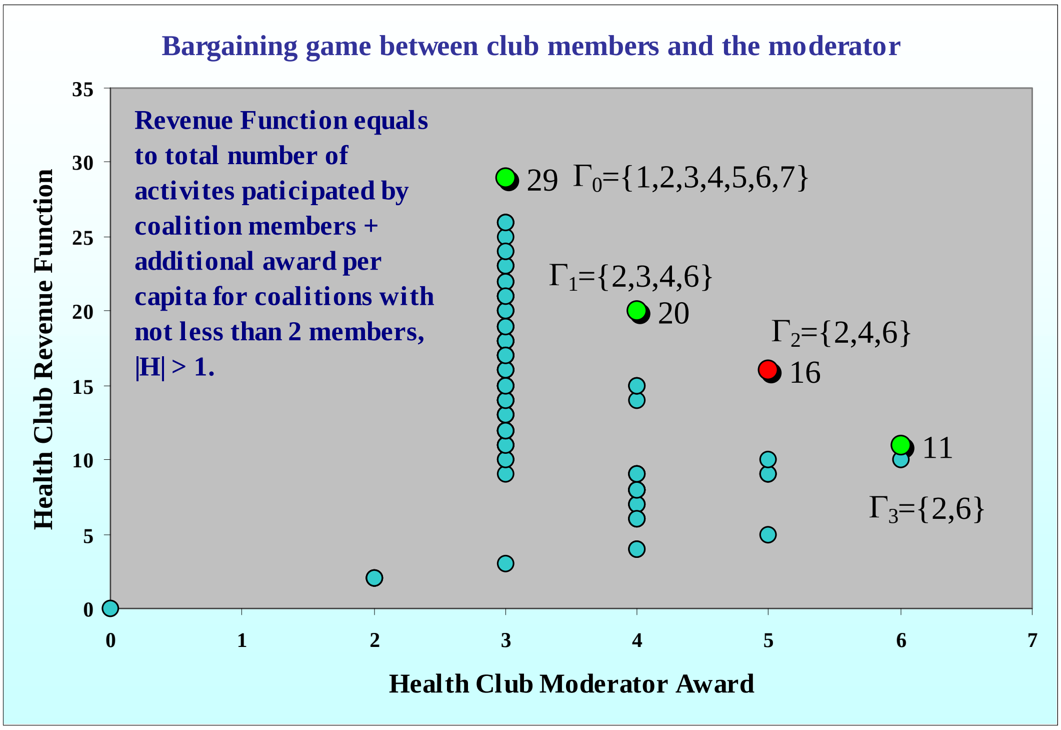

Figure 2.

N.B. Observe that utility pairs $(29,3)$, $(20,4)$, $(16,5)$ and $(11,6)$ constitute the Pareto frontier of bargaining solutions for bargaining problem between the moderator as bargainer nr.1 and coalitions as nr.2, accordingly, i.e., the grand coalition $N = \Gamma_0 = \{1,2,3,4,5,6,7\}$, three proper coalitions $\Gamma_1 = \{2,3,4,6\}$, $\Gamma_2 = \{2,4,6\}$ and $\Gamma_3 = \{2,6\}$. Solutions $(v(\Gamma_1) = 20, F(\Gamma_1) = 4)$ and $(v(\Gamma_2) = 16, F(\Gamma_2) = 5)$ maximize the product of players gains over the disagreement point $(0,0)$ at $20 \cdot 4 = 16 \cdot 5 = 80$, i.e., as we stated in the beginning of the paper, the solution might not be unique and some external consideration may help, for example, the sponsor expenses for $(20,4)$ which are equal to 24, while for $(16,5)$ expenses equal 21, might be decisive. That is the case, when the bargaining power $\theta = ½$ of the coalitions Γ_1, Γ_2 and the moderator are in balance. If not, choosing the coalition bargaining power $\theta < ½$, the moderator will be better off materializing the solution $(5,16)$. Coalition Γ_2 will be better off if $\theta > ½$.

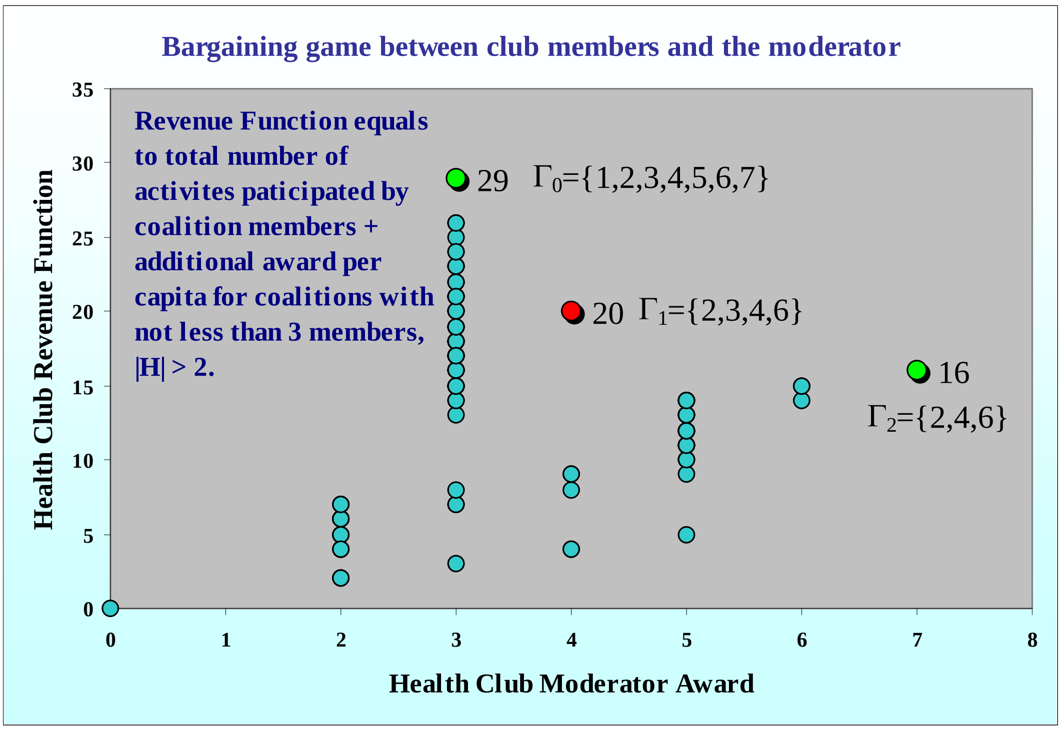

Figure 3.

N.B. Compare with Fig. 2 that coalition $\Gamma_3 = \{2,6\}$ lies no longer on the Pareto frontier.

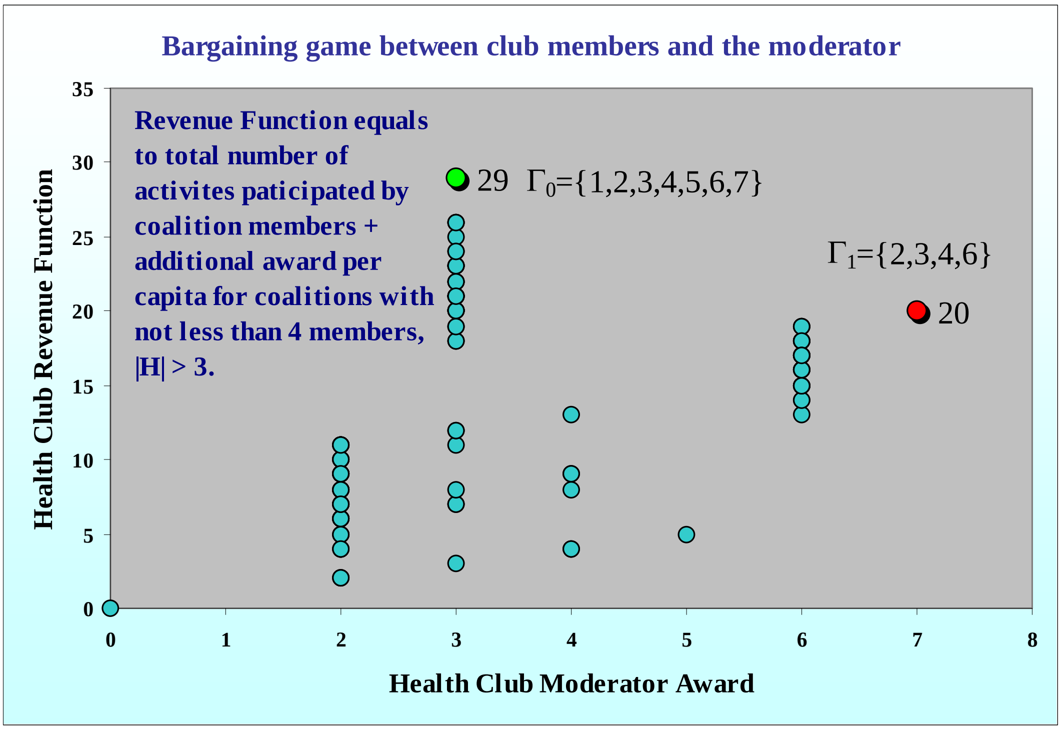

Figure 4.

N.B. Compare with Fig. 3 that coalition $\Gamma_2 = \{2,4,6\}$ lies no longer on the Pareto frontier.

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