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## **An Explorative Method to Study Markov Chain Structure**

### **I. Introduction.**

Current paper utilizes the theory of monotonic systems developed in [1] to the Markov chains. From one side, the interest called towards Markov chains originates from the reality that it is convenient to interpret a special class of absorbing chains as monotonic systems, while on other side an opportunity of illustration of monotonic system main properties exists for communication network<sup>1</sup> by example.

In order to disclose on conceptual level the technology developed in current paper of extracting extremal subsystems in Markov chains we bring about in slightly modified form an example of monotonic system – the communication network. Then we show in which way a Markov chain may be associated with the monotonic system and what principal operations may be performed on it towards utilization of monotonic systems theoretical apparatus described in [1].

In the paper [1] an example of some communication network has been considered under the pretext of a set  $W$ , which consists of communication lines/channels between some nodes – communicating units.<sup>2</sup> Suppose now, that each line has some redundancy

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<sup>1</sup> This term in the original paper was telephone switched net, which is not up to date anymore. It is not a long way to find “Switches” and its like, redirecting TCP/IP packages comparable to the telephone net.

<sup>2</sup> Switch is a natural mode to think about a node. Switch device can learn while the time pass where to address communication packages.

mechanisms build in, for example, the main and the reserved channels<sup>3</sup>. In case a direct line is not available between nodes in [1] the traffic might be organized through pass around channels. Above, in addition, a possibility of pass around communication is not excluded even in case there exists a direct channel.

In the example of paper [1] an average number of “denials” to establish the contact characterizes each pair of nodes was utilized. The number of denials usually characterizes the communication lines in the communication network.<sup>4</sup> In the model below and for the purpose of current investigation more convenient is to use a value inverse to the number of denials, and which will characterize the communication line throughput.

Suppose that each communication line (the main and the reserved channels) been characterized by the throughput  $c_{ij}$  or, in other words, by the maximum allowed bandwidth usage, for example in kilobytes. The value  $c_{ij}$  accounts the throughput of main and reserved channels. We then explicate the communication center  $s$  by the maximum allowed usage

$$c_s = \sum_{j=1}^n c_{sj}.$$

The traffic redirected through the node  $s$  along the main communications channel, as well as the reserved channel, between nodes  $s$  and  $j$  specifies thereupon a share of maximum allowed usage  $c_s$ . Observing an actual communication network the usage share must be lower than the maximum allowed share  $p_{sj} = \frac{c_{sj}}{c_s}$ . The usage share  $p_{sj}$  of the communication channel can be interpreted as a probability of a contact between the nodes  $s$  and  $j$ . If we suppose an identity of the main and the reserved channels then the quantity must satisfy an inequality

$$2 \cdot \sum_{j=1}^n p_{sj} < 1 \tag{1}$$

without exception for all  $s$ .

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<sup>3</sup> In practice the redundancy of a network may be guaranteed by some extra channels/lines activated only in urgent situations when the net usage exceeds some threshold.

<sup>4</sup> Network protocol analyzes can collect such kind of statistics.

Let a communication network, with pointed to above pass around traffic feasibility, functions during a long period of time by originating its main channels. We characterize using traffic along each main channel (to be more accurate the nodes  $i$  and  $j$ ) by the average number of hits  $\bar{p}_{i,j}$  in establishing direct, or might be indirect pass around contacts. Quite clear that numbers  $\bar{p}_{i,j}$  are slightly greater than  $p_{i,j}$ .

Suppose that somewhere in a channel malfunctioning happens. The change <sup>5</sup> in the communication network will be reflected in values of  $\bar{p}_{i,j}$  decrease. Now, it will be realistic to meet a higher usage of the network by activating a reserved channel. It is obvious as well, that in this case we can expect the increase in all  $\bar{p}_{i,j}$  values. This way organized communication network represents a monotonic system.

We encounter a problem: What kind of change malfunctioning/activating of a main/reserved channel interjects into numbers  $\bar{p}_{i,j}$ . For solving the problem it is necessary to explain it in Markov Chains vocabulary.

Consider a set  $W$  of communication channels described by a square matrix  $\|p_{i,j}\|_n^n$ . In case no channel exists  $p_{i,j} = 0$ . The well known common knowledge to be found in the theory Markov chains [2] tells us that such kind of matrices may be associated with a set of returning states for some absorbing Markov chain. In the vocabulary of such kind of chains we interpret the value  $\bar{p}_{i,j}$  as an average number of hits from node  $i$  into node  $j$  along the Markov chain while the malfunctioning of the main, or activating the reserved channels, will reflect itself in special formulas recalculating the average hits values  $\bar{p}_{i,j}$ . An action of type  $\ominus$  while an action of type  $\oplus$  in the vocabulary of monotonic systems is a malfunctioning and an activating of reserved channel accordingly.

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<sup>5</sup> For example, the OSPF (Open Short Path First) protocol automatically will redirect the traffic.

From all just said, our special class of absorbing Markov chains allows us to look at the problem how to differentiate the Extremal Subsystem of Monotonic System – the kernels, and the KFP procedure elaborated for this purpose in [1] actually accomplish the task for kernels search.

In section II of current paper we disclose the problem of kernel extracting on Markov chains. In the section III we show that result of  $\oplus$  and  $\ominus$  actions upon Markov chain entries in a transition matrix leads to Sherman-Morrison [3] (see the appendix) expressions for recalculating the numbers for average hits.

## **II. The problem of Kernel Extracting on Markov Chains**

Consider stationary Markov chain with finite number of states and discrete time. We denote the set of states by  $V$ . Stationary Markov chain can be characterized by the property that the pass probability from the state  $i$  to the state  $j$  in certain moment  $t+1$  does not depend upon which state  $s$  ( $s = 1, 2, \dots, n$ ) the considered chain arrived in  $i$  in the proceeding moment  $t$ . We denote by  $p(i, j, k)$  ( $p(i, j, 1) = p_{ij}$ ) the conditional probability of this pass from  $i$  to  $j$  within  $k$  units of time.

Below we consider only a special class of Markov chains that for arbitrary states  $i$  and  $j$  within some subset in  $V$  constrained by

$$\lim_{k \rightarrow \infty} p(i, j, k) = 0.$$

We know from the theory of Markov chains that this limit equals zero in case when the state  $j$  is returning, and as a consequence there must be some reversible states in such Markov chains. Without diminishing the generality of the consideration we will further consider the chains with only one reversible state, which must be at the same time an absorbing state.

The absorbing chains utilized below are the following:

1. There exist only one absorbing state  $\theta \in V$  ;
2. All remaining states are returning, and the probability for pass between the states in one step correspond to an entry in square matrix  $\|p_{ij}\|_n^n$  .
3. The probability for pass into absorbing state  $\theta$  from some returning state  $i$  in one step in accordance with 1 and 2 is equal to

$$p_{i\theta} = 1 - \sum_{s=1}^n p_{is} .$$

The monotonic System demands a definition of some positive and negative ( $\oplus$ ,  $\ominus$ ) actions upon system elements. For this purpose we make use of the average number of hits  $\bar{p}_{ij}$  from the state  $i$  into the state  $j$  along the chain [2]. It is known that the value of  $\bar{p}_{ij}$  is specified by the series

$$\bar{p}_{ij} = \sum_{k=1}^{\infty} p(i, j, k) . \quad (2)$$

The sufficient condition for series (2) to converge is at hand if the sum of entries in each row of the matrix  $\|p_{ij}\|_n^n$  is less than one. Further we consider elsewhere the chains fulfilling the conditions 1-3.

Let  $W$  be the set of all nonzero entries in the matrix  $\|p_{ij}\|$  . On the transition  $W$  set of the described above Markov chain we define following actions.

**Definition.** By action type  $\ominus$  on the element of the system  $W$  (nonzero element of the matrix  $\|p_{ij}\|$ ) we understand a decrease in its value by some  $\Delta p$  of its probability to pass in one step.

By analogy, we define the action  $\oplus$ . In this case the probability for pass in one step, which corresponds to the entry value  $p_{ij}$  is increasing by  $\Delta p$  . In case of some nonzero increment in the matrix  $\|p_{ij}\|$  element (from straightforward probability considerations) all averages numbers of hits  $\bar{p}_{ij}$  must increase either, but in case of  $\Delta p$  decrement, all

these average hits, also  $\bar{p}_{ij}$  values, must on the contrary decrease. In sum, introduced actions upon system  $W$  elements fully meet the monotonic condition [1], and system  $W$  arrange a monotonic system.

We must emphasize here that  $\Delta p$  changes in values of probabilities in one step within  $W$  are not specified in the definition of  $\oplus$ - and  $\ominus$ - actions upon the entries in the matrix  $\bar{p}_{ij}$ . Relatively rich possibilities exist for the change definition. For example, the increase (decrease) of each probability on a certain constant or the same change, but this time depending upon the probability value itself, etc. Making the definitions of  $\oplus$ - and  $\ominus$ - actions on absorbing Markov chain it is desirable to utilize authentic considerations. Below, by example of communication network, we describe one of such considerations.

Let  $W$  be the set of all possible transitions in one step among all returning states of absorbing chain. These transitions in the set  $W$  keep up a correspondence with nonzero elements of the matrix  $\|p_{ij}\|$ . Let  $T$  be a certain subset of the set  $W$ , which points at nonzero elements as considered. Denote by  $p(T, i, j, k)$  the probability that the chain passes from the state  $i$  into the state  $j$  within  $k$  time units constrained by the condition that during this period the passes in one step upon the set  $T$  have been changed by our  $\oplus$ - or  $\ominus$ - actions. This condition corresponds to that the passes along the set  $W \setminus T \equiv \bar{T}$  proceed in accordance with the “old” probabilities, while along  $T$  in accordance with the “new”. We do not exclude the case when no  $\oplus$ - or  $\ominus$ - actions been implicated – the set  $T = \emptyset$ . In this case we omit the  $T$  symbol notation in the corresponding probabilities.<sup>6</sup>

The average numbers of hits from  $i$  into  $j$  subject to constraint that some passes in the set  $T$  been changed by actions is specified by a series

$$\bar{p}(T, i, j) = \sum_{m=1}^{\infty} p(T, i, j, m). \quad (3)$$

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<sup>6</sup> We suppose that actions do not violated the convergence of probability series, see condition (1).

Let us now turn to the collections of weights specified by a monotonic system  $W$ . We define a collection  $\Pi^+H$  on the subset  $H \in W$  as a collection of real numbers  $\{\bar{p}(\bar{H}, i, j) | (i, j) \in H\}$  in case the positive  $\oplus$ - actions occur on  $\bar{H} = W \setminus H$ , and the collection  $\Pi^-H = \{\bar{p}(\bar{H}, i, j) | (i, j) \in H\}$  when the negative  $\ominus$ - actions occur.

In the paper [1] we have proved that in monotonic system necessarily two kinds of subsystems exists: The  $\oplus$  and  $\ominus$  kernels. The above introduced definitions of average hits numbers  $\bar{p}(\bar{H}, i, j)$  allows us to formulate the notion of  $\oplus$  and  $\ominus$  kernels in the Markov chain.

**Definition.** By the Extremal Subsystem of passes on absorbing Markov chain – the  $\oplus$  and  $\ominus$ -kernels we call a system  $H^\oplus \subseteq W$ , on which the functional

$$\max_{(i,j) \in H} \bar{p}(\bar{H}, i, j) \quad (4)$$

reaches its global minimum on  $2^W$ , and  $\ominus$ -kernels will be a subsystem  $H^\ominus \subseteq W$  where the functional

$$\min_{(i,j) \in H} \bar{p}(\bar{H}, i, j) \quad (5)$$

reaches its global maximum as well.

We will now pay attention to the introduced notions of  $\oplus$  and  $\ominus$  kernels illustration by example on communication network described in the beginning.

The probabilities of hits  $p_{i,j}$  (without pass, i.e., in one step) between nodes  $i$  and  $j$  ( $i, j = 1, 2, \dots, n$ ) allow us to construct for communication network an absorbing chain satisfying the conditions 1-3 above. In fact, as we already pointed to, only one condition is mandatory regarding the inequality (1), which is a natural condition for considered communication network. Condition 2-3 can be guaranteed by the Markov chain design. In this case numbers  $p_{i,j}$  may be interpreted as probabilities of pass in one step, and the number  $\bar{p}_{i,j}$  as average number of hits from  $i$  into  $j$ , might be with the help of indirect pass around along other lines in chain.

The search of  $\oplus$  and  $\ominus$  kernels on actual Markov chain, reconstructed from communication network, demands an perceptible definition of  $\oplus$  and  $\ominus$  actions. In the beginning we noticed that  $\ominus$  action might be a malfunctioning in the main channel, and  $\oplus$  action might be the reserved channel activating. On the Markov chain the malfunctioning turns up as null stilling the corresponding probability while the activating of reserved channel as doubling of its initial probability value.<sup>7</sup> The condition (1) guarantees that in any circumstances of such  $\oplus$  and  $\ominus$  actions the convergence of series (2) and (3) will not be violated.

We suggest a noticeable interpretation of  $\oplus$  and  $\ominus$  kernels in Markov chain as follows starting from the Markov chain characteristics introduced here in terms of communication network.

In Extremal Subsystem  $H^\oplus$  all communication lines/channels remain without changes, but in all lines outside  $H^\oplus$  their reserved channels been activated. The extremal value of the functional (4) shows that the average number of hits within channels belonging to  $H^\oplus$ , including the indirect pass around of hits (indirect hit uses at least 2 steps to reach the destination), is relatively low. This means that the lines within  $H^\oplus$  kernel are “immune” with the respect of package deliveries malfunctions – most of the traffic packages pass along direct lines. The set of lines in  $H^\ominus$  kernel comprises an opposite property. The main channels in  $H^\ominus$  kernel are most “appropriate” in organizing a “high-quality” indirect communications, but sensible for malfunctions “snowballing” or “bag wagon” effects: Along  $H^\oplus$  the indirect communication is hampered most for some reason.

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<sup>7</sup> We stress once again that  $\oplus$  and  $\ominus$  actions are subjective evaluation in actual situation.

### III. Monotonic System Weight Functions on Markov Chains

In section II we defined some  $\oplus$  and  $\ominus$  actions upon the transition matrix entries in one step corresponding to returning states. In current section we will develop an apparatus, which allows us to account the changes induced by these two types of actions into the averages of hits numbers from one returning state  $i$  into the other state  $j$ . We describe here and deduce some tangible weight functions intended to be used in concurrence with our formal description of monotonic system following the conventions from [1]. Let recollect first the notion of weight function before we give an account regarding the main section contents.

Suppose that in the system  $W$ , which in the case of Markov chain is characterized as a collection of entries in matrix  $\|p_{ij}\|_n^n$  corresponding to passes among returning states, a subset  $H$  has been extracted: The set  $H$  consists of one step transitions. As a result of successive actions of type  $\ominus$ , one by one process (see section II), upon the elements in  $\bar{H}$  ( $\bar{H}$   $j$  – complementary of  $H$  to  $W$ ) one can establish the average numbers of hits within the transition set  $H$  – the weight system  $\Pi^-H$ . By analogy on the set  $\bar{H}$  a succession of  $\oplus$ -actions establish the weight system  $\Pi^+H$ . The average hits number in the vocabulary of section II may be written as  $\bar{p}(\bar{H}, i, j)$  – the limit values for series (2) on nonzero elements for transition matrix  $P$  corresponding to the entries/lines within the set  $H$ . Further we will call the numbers  $\bar{p}(\bar{H}, i, j)$  the weight functions.

Let establish now the general form of the weight functions on Markov chains as matrix series. Such form of matrix series establishment can explain the mechanism of the defined in the section II actions upon the elements of monotonic system – the Markov chain.

The weight function on Markov chain may be found using the series (2), where the single element  $(i, j)$  in the series is the probability of chain pass from  $i$  into  $j$  constrained by the condition that upon the set  $\bar{H}$  actions have been performed.

The general matrix form of such transition probabilities described in section II is as follows:  $\theta$

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ p_{1\theta} & & & \\ \cdot & P & & \\ p_{n\theta} & & & \end{pmatrix} \quad (6)$$

where  $\theta$  – absorbing state of the chain;

$p_{i\theta}$  – the probability of pass from the  $i$ 's returning state into the absorbing state  $\theta$  ;

$P$  – the transition matrix of probabilities between the returning states within one step, matrix dimension  $n \times n$ .

Using Chapman-Kolmogorov equations [2] the element  $p(T, i, j, m)$  in series (3) may be found as the  $m$ -s power of the matrix (6), and hereby it occupies an entry in the matrix  $P^m$ .

In summary, the collection of series (3) may be written as the matrix series

$$\bar{P}_T = I + P_T + P_T^2 + \dots,^8 \quad (7)$$

$P_T$  – the matrix, where upon nonzero elements within the set  $T$  type  $\oplus$  and  $\ominus$  actions been performed. Recall that in the definition of monotonic system the weight function on the set  $H \subseteq W$  take advantage only of complementary set  $\bar{H}$  to the set  $H$  : The set  $\bar{H}$  is actually the set of performed actions. Given the elements of the set  $W$  as matrix entries  $\bar{P}_{\bar{H}} = \|I - P_{\bar{H}}\|^{-1}$  the latter matrix is the weight functions collection on the Markov chain

– the same as the matrix limit of (7).

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<sup>8</sup> We suppose that  $p(T, i, j, 0) = \delta_{ij}$  what the unity matrix  $I$  highlights. In the vocabulary of Markov chains [4] theory we call matrices of type  $P_T$  the fundamental matrices.

In the vocabulary of fundamental matrices the actions upon the monotonic system elements are the transformations, one by one, in succession, from the matrix  $\|I - P_T\|^{-1}$  to the matrix  $\|I - P_{T \cup \alpha}\|^{-1}$ . Calculus of such a transformation is quite “hard operation”. In order to organize the search of  $\oplus$  and  $\ominus$  kernels on the basis of constructive procedures (KFP) described in [1] the utilization of matrix form is inappropriate. To extract the extremal subsystems on Markov chains successfully and make the most of the developed theory of monotonic system a more effective technology is needed, which leads us to Sherman-Morrison relationships [3].

The solution to account the changes emerging as the result of  $\oplus$  and  $\ominus$  actions upon the transition matrix elements within one step in the fundamental matrix of Markov chain may be archived in the following manner. Suppose that instead of old probability  $p_o$  pass in between the returning states  $i$  and  $j$ , an updated – a new probability  $p_n = p_o + \Delta p$  has been inserted, where the action ( $\pm \Delta p$ ) – an increment/decrement – is assumed. In case of ( $+\Delta p$ ) we have the  $\oplus$ -action, in case of ( $-\Delta p$ ) – the  $\ominus$ -action. The change induced by one of these action may be treated as two successive changes: The probability  $p_o$  is replaced by 0 and the replacement recalculated, thereafter the transition probability is reestablished with the new value  $p_n$  and the change in the fundamental matrix is recalculated immediately after the first recalculation.

The relationships accounting the changes in the fundamental matrix  $\overline{P}_T$  as a result of element  $\alpha$  null stilling in the matrix  $P_T$  and the relationships for accounting the changes in  $\overline{P}_T$ , also in the reverse case of  $\oplus$ -actions, may be found in the Appendix I.

In sum, for the search of extremal subsystems following the theory of constructing the defining sequences on system  $W$  elements with the aid of KFP procedures in [1], it is necessary to obtain some well-organized and distinct recurrent expressions, which can account the changes in the matrix  $\overline{P}_T$  to be transformed to the matrix  $\overline{P}_{T \cup \alpha}$ . Such like formulas for specified  $\Delta p$ , which allow us to transform from  $\overline{P}_T$  in order to find the matrix  $\overline{P}_{T \cup \alpha}$  are worked out in the Appendix II on the basis of the expressions II 1.3 and II 1.4.

With the aid of these recurrent expressions in the Appendix II it is possible to obtain on each set  $H \subseteq W$  the collection of weights  $\Pi^+H$  or  $\Pi^-H$  performing the successive implementation of expressions II 2.5 to all elements upon the set  $\bar{H}$ . These expressions II 2.5 mirror the transformation of system elements weights  $\pi$  into  $\pi_\alpha$  in view of theoretical apparatus of monotonic systems [1]. Indeed, we construct the collection  $\Pi^+H$  in case the  $\Delta p > 0$ , but if  $\Delta p < 0$  we construct the collection  $\Pi^-H$ .

### Appendix I

Consider the value  $\bar{p}(T, i, j)$  produced by the series (3). Each component of this series may be treated as the measure of all passes in  $m$  step of time units initiating from  $i$  and ending in  $j$ . This assemblage of transitions is a union of two nonintersecting collections: first set – the passes from  $i$  to  $j$  with mandatory transition, at least once, along  $\alpha \in W$ , the second – the set of passes from  $i$  to  $j$  avoiding this transition  $\alpha$ . Each passage from the first set consists of two passes: a pass avoiding  $\alpha$  being in  $t$  steps long, and a pass in  $m - t - 1$  steps of time units, but passing along  $\alpha$ . In other words: the passages in  $t$  steps avoid the pass along  $\alpha$ , but passages in  $m - t - 1$  steps use this pass  $\alpha$ .

We introduce the following notation:  $\bar{p}(T^0, i, j, k)$  – the average number of hits from  $i$  into  $j$  with the transition matrix  $P_T$ , where the nonzero element  $\alpha$  is null stilled,  $p(T^0, i, j, k)$  the probability of transition consequently – this time without  $\alpha$ .

Implementing the introduced notification we get

$$p(T, i, j, m) = p(T^0, i, j, m) + p_\alpha \cdot \sum_{t=0}^{m-1} p(T^0, i, \alpha_b, t) \cdot p(T, \alpha_e, j, m - t - 1) \quad \text{II 1.1}$$

$$p(T, i, j, m) = p(T^0, i, j, m) + p_\alpha \cdot \sum_{t=0}^{m-1} p(T, i, \alpha_b, t) \cdot p(T^0, \alpha_e, j, m - t - 1) \quad \text{II 1.2}$$

where  $\alpha_b$  – the state, from where in one step the pass begins, and  $\alpha_e$  where it ends;

$p_\alpha$  – the pass along  $\alpha$  in one step; corresponds to the element  $\alpha$  of the matrix  $P_T$ .

The first component in II 1.1 and II 1.2 brings in the value of  $p(T, i, j, m)$  – the measure of transitions avoiding the pass along  $\alpha$ , and components standing under the sum sign represent the probability that the state  $\alpha_b$  (for the relationship II 1.1) and  $\alpha_e$  (for the relationship II 1.2) been reached by the first and the last pass along  $\alpha$  in the moments  $t$  and  $t + 1$  accordingly.

Let calculate the values  $\bar{p}(T, i, j)$  using the relationship II 1.1. We conclude after summarizing each of the equations II 1.1 from 1 to  $M$  and thereafter changing the order of sums in our double sum that

$$\sum_{m=1}^M p(T, i, j, m) = \sum_{m=1}^M p(T^0, i, j, m) + p_\alpha \cdot \sum_{t=0}^{M-1} p(T^0, i, \alpha_b, t) \cdot \sum_{s=1}^{M-t} p(T, \alpha_e, j, s-1)$$

Dividing both parts of the latter equation on

$$\sum_{t=0}^{M-1} p(T^0, i, \alpha_b, t),$$

then we conclude from the theorem of Norlund averages [2] considering the sequence  $a_t = p(T^0, i, \alpha_b, t)$  and  $b_{m-t} = \sum_{s=1}^{M-t} p(T, \alpha_e, j, s-1)$  and meanwhile increasing

$M \rightarrow \infty$  that for the sequences  $a_n$  and  $b_n$  following relations are valid:

$$\bar{p}(T, i, j) = \bar{p}(T^0, i, j) + p_\alpha \cdot \bar{p}(T^0, i, \alpha_b) \cdot \bar{p}(T, \alpha_e, j). \quad \text{II 1.3}$$

Analogous relationship can be deduces exploiting the composition II 1.2, exactly:

$$\bar{p}(T, i, j) = \bar{p}(T^0, i, j) + p_\alpha \cdot \bar{p}(T, i, \alpha_b) \cdot \bar{p}(T^0, \alpha_e, j). \quad \text{II 1.4}$$

## Appendix II

We introduce following notifications. Let  $\bar{p}(T_o, i, j)$  is the matrix  $\bar{P}_T$  element, and  $\bar{p}(T_n, i, j)$  is the matrix  $\bar{P}_{T \cup \alpha}$  element. Let rewrite II 1.3 and II 1.4 with respect to these notifications, what yields to

$$\bar{p}(T_n, i, j) = \bar{p}(T^0, i, j) + p_n \cdot \bar{p}(T^0, i, \alpha_b) \cdot \bar{p}(T_n, \alpha_e, j); \quad \text{II 2.1}$$

$$\bar{p}(T_o, i, j) = \bar{p}(T^0, i, j) + p_o \cdot \bar{p}(T_o, i, \alpha_b) \cdot \bar{p}(T^0, \alpha_e, j). \quad \text{II 2.2}$$

From the relationships II 2.1 and II 2.2 it follows that the new value for the average hits from  $i$  into  $j$  is equal to

$$\begin{aligned} \bar{p}(T_n, i, j) = & \bar{p}(T_o, i, j) + \\ & p_n \cdot \bar{p}(T^0, i, \alpha_b) \cdot \bar{p}(T_n, \alpha_e, j) - p_o \cdot \bar{p}(T_o, i, \alpha_b) \cdot \bar{p}(T^0, \alpha_e, j) \end{aligned} \quad \text{II 2.3}$$

Substituting in II 2.1 the state  $i = \alpha_e$ , we get that

$$\bar{p}(T_n, \alpha_e, j) = \bar{p}(T^0, \alpha_e, j) / (1 - p_n \cdot \bar{p}(T^0, \alpha_e, \alpha_b)),$$

and from II 2.2 with the same  $i = \alpha_e$  we get

$$\bar{p}(T^0, \alpha_e, j) = \bar{p}(T_o, \alpha_e, j) / (1 + p_o \cdot \bar{p}(T_o, \alpha_e, \alpha_b)).$$

Replacing the latter expression into the preceding, and taking into account that

$$\bar{p}(T^0, \alpha_e, \alpha_b) = \bar{p}(T_o, \alpha_e, \alpha_b) / (1 + p_o \cdot \bar{p}(T_o, \alpha_e, \alpha_b)),$$

we get at last that

$$\bar{p}(T_n, \alpha_e, j) = \bar{p}(T_o, \alpha_e, j) / (1 - \Delta p \cdot \bar{p}(T_o, \alpha_e, \alpha_b)). \quad \text{II 2.4}$$

The expression II 2.1 is valid if we replace  $T_n$  by  $T_o$  and  $p_n$  by  $p_o$ , and if in the expression II 2.2 we make the reverse replacement. Substituting  $j = \alpha_n$  in the expression II 2.2 but first regrouping it by this reverse replacement what yields to

$$\bar{p}(T^0, \alpha_e, j) = \bar{p}(T_o, \alpha_e, j) / (1 + p_o \cdot \bar{p}(T_o, \alpha_e, \alpha_b)).$$

Finally, we deduce the expression for accounting the changes in the fundamental matrix  $\bar{P}_T$  by simplifying the last two equalities and the expression II 2.4 after collecting subexpressions and making rearrangements to transform  $\bar{P}_T$  into the matrix  $\bar{P}_{T \cup \alpha}$ . In standard vocabulary of section III the ultimate expression form is as follows:

$$\bar{p}(T \cup \alpha, i, j) = \bar{p}(T, i, j) + \Delta p \cdot \frac{\bar{p}(T, i, \alpha_b) \cdot \bar{p}(T, \alpha_k, j)}{1 - \Delta p \cdot \bar{p}(T, \alpha_e, \alpha_b)}. \quad \text{II 2.5}$$

### Literature

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