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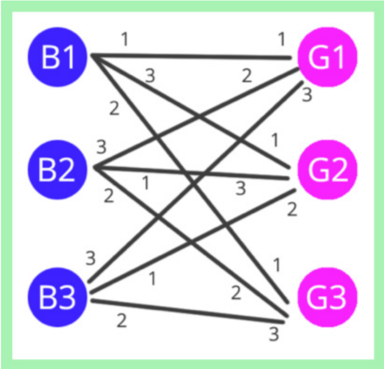

Partial Matching in the Marketing Game: Reassessing Incompatibility Indicators

Abstract. The game under scrutiny serves as a sophisticated model mirroring the intricacies of real-world scenarios within marketing agencies, where the allocation of clients among employees undergoes a continuous series of assessments and prioritizations. This allocation process, termed "matching" in economic discourse, unfolds through a sequential chain of reflections, with each decision influencing subsequent steps. However, the dynamic nature of this environment can result in mismatches between clients and employees, leading to marketing instability. To mitigate this instability and address the inherent fuzziness in marketing, we propose employing indicators of inadequacy or incompatibility to identify when an employee may not be the best fit for a client. By regularly reassessing these metrics throughout the marketing process, our aim is not merely to minimize failures but to optimize outcomes and minimize compensation requirements.

Keywords: marketing game; core; rational choice; cooperation; matching
JEL classification: C71; C78

1. Introduction

Roommate problem [1] proposed by Gale and Shapley in 1962 was also considered by Bergé (1958, [2]) and has since become the canon for various forms of economic stability. The canonical solution assumes a complete agreement or grand matching for all the members of economic community consisting of an even number of agents. One of the difficulties that we have encountered is expressed in the triplicity of the interests of the clients and marketing agency. Yes, it is true that both clients and employees of an agency have individual interests that may sometimes conflict. It is also true that in any organization, the interests of the staff as a whole can also arise. Even more confusing, however, is the paired interest of clients in what we have called the "*marketing game*." The dynamic maneuvering approach proposed by Lefebvre and Smolyan in 1968, [3], can indeed be a useful tool in situations dealing with many players, which have competing interests in a dynamic and multi-stage marketing environment. The quasi-core concept presented, referring to a partial matching that is optimal for all parties involved, can provide a valuable basis for achieving mutually beneficial results. In a complete match situation, both parties may be satisfied with the outcome, but more often than not, there will be areas of inconsistency. In such cases, partial matching can be useful in determining solutions that are optimal for everyone, although not necessarily ideal. Indeed, the judgment was made that "*the best old client is still the best.*"

We refer to partial stability in which the “rewards and compensations” paid to clients and agency cannot be increased further, despite attempts to improve the situation. The partial stability in this scenario indicates that marketing reservations has reached an optimal state, and further changes or attempts to improve the situation will result in negative effects for the clients and the agency. The quasi-core concept refers to a solution that is considered to be stable, but not necessarily optimal in the game theory context. The situation we are referring to is known as a "forbidden set". In the recent articles by Richter and Rubinstein [4] suggest that there may be a set of matches "Z" that cannot be realized. This can result in ending the game prematurely, similar to what can happen in a university environment during the early years of higher education.

Indeed, soon after the start of their studies, many university and college students are trying to change the nature of their studies or prefer other tasks. Students, in their own opinion, may choose incompatible educational programs, while the composition of the students themselves in a particular program may also not be optimal. Students and programs may not be compatible with each other. So-called discrepancies in mutual rankings have been one of the reasons (Leo Võhandu, LV, 2010, [5]) for the unacceptably high percentage of Estonian students who drop out in their first years of study, wasting their time and entitlement to state support. However, a better matching between students and educational programs can mitigate this problem.

The problem being discussed is a variation of the stable matching (Bergé; Roth & Sotomayor, 1990, [6]), where the goal is to match pairs of agents (in this case, students and programs) in a way that satisfies certain preferences while avoiding blocking pairs. To solve the problem, it was proposed to introduce a "coincident total" as the sum of "matching rankings" selected in two directions—the horizontal rankings of students involved with programs and the vertical rankings of programs matching to students. According to LV, the best solution among all possible horizontal and vertical sums of rankings is where the sum reaches its minimum.

Finding the "coinciding minimum" is a difficult task. Instead, LV suggested a Greedy type workaround. According to LV, the best solution to the problem of matching students and programs would be a fairly close (cf. Cormen et al, 2001, [7]) accumulation of the sum when moving along the direction of mutual matching in a non-decreasing order of rankings. Apparently, the approach of LV to the solution of the problem was drawn up in the spirit of classical utilitarianism, when the sum of utilities should be maximized or minimized (Bentham, Principles of morality and legislation, 1789, [8]; Sidgwick, Methods, Ethics, London, 1907, [9]). The reader studying matching problems may also find information on these issues, where a number of ways to construct an optimal matching strategy have been discussed (Veskioja, 2005, [10]).

The setting of the marketing game will be presented with an attempt to explain by an example a rather complex intersection of interests, where readers must be prepared to engage in a reality masquerade in order to understand the basic concept of the coalition game (Gillies, 1959, p. [11]; also noted as the core by John von Neumann and Morgenstern, 1953, [12]). In particular, we hope to shed light on the dynamic or multi-stage nature of client and agency staff ranking re-evaluation during the game. It should be emphasized that although the game primitives are a separate mathematical entity in a completely different context, we "borrowed" the idea of LV-s ranking to evaluate the rewards of matching. For this reason, we have changed the nomenclature of payments for mutually incompatible agreements, i.e., "imputation", or "imputed compensations" in order to introduce a payment scale that has a monotone character. The scale is organized as incompatibility indicators in the form of a "Monotone System."

N.B. The Monotone system (MS, see also "Monotonic Link Functions", Seiffarth et al, 2021, [15]) is used to reassess the risk indicators of entering into agreements that are not compatible by considering the mutual rankings of clients and employees. The indicators have a monotonic property, which allows for dynamic adjustments to be made in response to changes in rankings ensuring that they remain in synchrony. The system implements the concept of partial matching by ordering the indicators through a process caused by the inclusion of subsets taken from a general set of indicators. The Monotone system formalizes and generalizes the concept of ordering, sequencing, or arrangement of elements in subsets, providing a structured and systematic approach to assessing incompatibility risk in various contexts. The theory was initiated by Mullat (1971, [16]) and since then was further developed and published in Russian periodical of MAIK in 1976. Plenum Publishing Corporation originally distributed it in English. Without the use of the MS, the analysis of marketing game scenarios would be limited and potentially inaccurate, as the system provides a clear framework for understanding the relationships between parties on marketing platform and their impact on each other. Perhaps Monotone Systems provide a framework for analyzing the properties of specific multi-stage dynamic games.

Roadmap. The rest of the paper will be structured as follows. In Section 2, the primitives and notations used in the paper are explained. Section 3 endows with a detailed explanation of the marketing game and its analysis, including basic definitions and the non-traditional theory of quasi-core stability. The main body of the paper ends with Section 4, which contains the conclusions and suggestions for future work. The Appendix provides additional information and computational algorithms for the reader to better understand the concepts discussed in the paper. This includes the explanation of a claim that "*the best old client is still the best*" in A1; compatibility indicators reassessment algorithms in A2; the basic concept of canonical stability in A3; and computational algorithms visualization in A4, A5, and A6. An Excel spreadsheet is also provided to help with the technical details.

2. THE GAME PRIMITIVES AND NOTATIONS

We use a two-sided marketing platform, where clients and agency staff both play an active role in the matching process. The game is played in discrete time slices or reflections k, with an increasing k as the game progresses through the periods. Clients and employees of the agency enter into contracts or deals

α_k during the period k after which the products and services of the agency prescribed in contracts are considered reserved. It is assumed that the participants may enter into agreements or matches that are not well suited for them, or that may not be compatible with other agreements or matches they have made. This can create dynamic changes in the willingness of participants to enter into agreements or matches describing a multi-stage reflexive process in which the list of matches expands and the list of potential opportunities is gradually narrowed down over time. The game can end at any point at the request of clients or the marketing agency, and it can end with a partial matching or a complete match.

Having said that, we are talking about matchmaking or partnering event or activity where participants are matched based on compatibility. If no participants have been able to find a suitable partner, then it may be difficult to continue offering rewards or compensations. In such a scenario, the marketing agency staff and clients may need to re-evaluate their approach and criteria for matching participants, or consider whether to continue the event at all. It is important to carefully consider the potential risks and drawbacks of offering compensations in situations where the results of the matchmaking process are uncertain or unreliable. Ultimately, the decision on whether or not to continue the marketing effort should be based on a careful assessment of the risks and benefits involved.

2.1. Visualization example

Five clients and five employees decided to attend the marketing game. Clients will be asked to prioritize employees; while agency staff will be asked to prioritize eligible clients accordingly from the agency's point of view. This information to match clients with eligible employees and vice versa employees with clients will be used to reassess indicators of incompatibility. Clients and agency staff providing this information have been promised to collect boxes of "*goodies*" and are henceforth referred to as *participants*, while others are marked as *blanks* by default and cannot participate in the game.

Game participants who find a suitable partner will be rewarded, while failing that receive compensations or cheering payoffs for bad luck. In order to cover the expenses, the marketing fee is set at -50€ per participant. Thus, the amount of +500€ will be at the disposal of the cashier. The tables $W = \|w_{i,j}\|$ and $M = \|m_{i,j}\|$, Table-1&2, are used to represent the dynamic ranking of clients and marketing agency, respectively; also known as strict ranking or linear order. There are $\{1,...,i,...5\}$ clients and $\{1,...,j,...5\}$ staff employees. The $w_{i,j}$ cells indicate clients $i = \overline{1,5}$ who revealed their rankings positioned in the rows of table W towards employees as horizontal permutations of numbers $\langle 1,2,3,4,5 \rangle$. Similarly, agency staff revealed its idea on clients ordering in $m_{i,j}$ cells $j = \overline{1,5}$, as vertical permutations in the columns of table M, relative to the employees. The numbers $\langle \overline{1,5} = 1,2,3,4,5 \rangle$ can be repeated in the columns of table W and in the rows of table M. More than one client may prefer the same employer at priority level $\mathrm{w}_{\mathrm{i,j}}$. Multiple employees, accordingly, may be well suited to the same client at the level $\mathrm{m}_{\mathrm{i,j}}$ by the staff reflexive idea. When rankings have been revealed, they can form two 5×5 tables, resulting in $2 \times 5 \times 5$ combinations. The table $\mathrm{R} = \left\| \mathrm{r}_{\mathrm{i,j}} \right\|$, Table-3, shows the matching of clients and employees, who have mutual risks $\mathrm{r}_{\mathrm{i,j}} = \mathrm{w}_{\mathrm{i,j}} + \mathrm{m}_{\mathrm{i,j}}$ of incompatible agreements.

		E_1	E_2	E_3	E_4	E_5			E_1	E_2	E_3	E_4	E_5			E_1	E_2	E_3	E_4	E_5
	L_1	1	5	3	2	4		L_1	3	4	2	1	2		L_1	4	9	5	3	6
	L_2	5	4	1	2	3		L_2	1	3	4	2	4		L_2	6	7	5	4	7
W	L_3	3	5	4	2	1	$+\mathrm{M}$	L_3	5	2	3	4	3	$=\mathrm{R}$	L_3	8	7	7	6	4
	L_4	2	5	3	1	4		L_4	4	5	1	3	1		L_4	6	10	4	4	5
	L_5	4	3	1	2	5		L_5	2	1	5	5	5		L_5	6	4	6	7	10
k=1	Clients' Reflexive Priorities							Staff Reflexive Priorities							Initial Incompatibility					

2.2. Rawlsian postulate and compensations' arithmetic

The Rawlsian postulate argues that *"institutions"* should be organized in such a way as to benefit the least advantaged members of community: *"The welfare of the worst-off individual is to be maximized before all others, and the only way inequalities can be justified is if they improve the welfare of this worst-off individual or group..."* Public Choice III, D.C. Mueller, p.600, [14]. Based on this postulate, players may have the following ideas of how the game can continue.

Indeed, let the compensations sums, even if this is impractical postulate, are set proportionally to $\frac{1}{2}\mathrm{r}_{\mathrm{i,j}} \times 10$€; in such a case, the participants profit can reach 50€ for free! Instead, we try to design the game by encouraging clients and agency employees to follow Rawls' "high of the least" second principle of justice [13]. Some of participants signed deals, while others tend to dynamically reassess the risks $\mathrm{r}_{\mathrm{i,j}}$ of incompatible agreements. These lucky dealers $\sigma = [\boldsymbol{i}_\sigma, \boldsymbol{j}_\sigma]$, or $\sigma = [\mathrm{L}_\sigma, \mathrm{E}_\sigma]$, were promised rewards. Unsuccessful participants, those who have not yet signed a deal given that only matchings with high level of mutual risks $\mathrm{r}_{\mathrm{i,j}}$ remained, can claim compensations. On initial reflection, let the expected rewards of all participants are proportional to $\min_{\mathrm{i,j}=\overline{1,5}} \mathrm{r}_{\mathrm{i,j}}$. In Table 3, the lowest mutual risk is $\mathrm{r}_{1,4} = 3$. All participants in the game are paid 10€ for goodies if the game ends immediately; in the opposite situation, when the game continues until the complete matching—the situation is the same—the participants still lose, in contrast to the partial matching. The losses of all participants in both cases will be –40€. We assume that participants in the pair $\sigma = (\boldsymbol{w}_1, \boldsymbol{m}_4)$ receive $\boldsymbol{w}_1, \boldsymbol{m}_4$ =+30€ each, since by Rawls' principal rule, the *argmin* $\mathrm{r}_{1,4} = 3$, viz., 3×10€ = +30€. The other 8 participants, now according to the compensation rule, will receive half, $\frac{1}{2} \cdot \mathrm{r}_{1,4} \times 10$€ =

+15€. Everyone benefits from the matching $\sigma=[1,4]$. Indeed, partners $[1,4]$ will be able to reduce their losses $\boldsymbol{w}_1$, $\boldsymbol{m}_4$ to –10€, since their gain according to the rules of the game will be $+30€=r_{1,4}\times 10€$. Considering the +10€ cost of goodies, other 8 participants will also reduce their losses, but only to –25€, since –50€ was paid as an entry fee will be reduced by +15€ from the compensation sums.

What happens if the participants $\sigma=[1,4]$ decide to sign the agreement in the initial time period $k=1$? The entire table R must be dynamically reassessed into sub-block X to reflect that participants $[1,4]$ have been matched. Indeed, the clients $\{2,3,4,5\}$ and the agency staff employees $\{1,2,3,5\}$ can no longer rely on their latent partners $[1,4]$. The ranking's scale $\langle 1,2,3,4,5\rangle$ is narrowed dynamically to $\langle 1,2,3,4\rangle$, which leads to a decrease in risks $r_{i,j}$.

To reflect this, Table 1–3 have been reassessed to Table 4–6. The yellow cells determine the sub-block $X=R\div\sigma$; $C(X)=\{\arg\min\alpha\in X\}$ determine the green cells choice operator, where the partners $\sigma=[1_\sigma,4_\sigma]$:

		E_1	E_2	E_3	E_4	E_5			E_1	E_2	E_3	E_4	E_5			E_1	E_2	E_3	E_4	E_5
	L_1							L_1							L_1					
	L_2	4	3	1		2		L_2	1	3	3		3		L_2	5	6	4		5
$\mathcal{W}$	L_3	2	4	3		1	$+\mathcal{M}$	L_3	4	2	2		2	$=X$	L_3	6	6	5		3
	L_4	1	4	2		3		L_4	3	4	1		1		L_4	4	8	3		4
	L_5	3	2	1		4		L_5	2	1	4		4		L_5	5	3	5		8
k=2	Clients' Reflexive Priorities							Staff Reflexive Priorities							Reassessed Incompatibility					

The compensation sum has not changed, and is still +15€. The balance $-50€+10€+2\times 15€=-10€$ of the pair $[1,4]$ improves; L_1, E_4 each receive, $\boldsymbol{w}_\sigma, \boldsymbol{m}_\sigma$ =+30€, $\sigma=(\boldsymbol{w}_1,\boldsymbol{m}_4)$ as rewards for matching based on the rule that it is equal to twice of the minimum compensation. For those not yet matched, the individual balance remains negative, viz., –25€.

Inclusive goodies, the cashier balance 500€–2×(10€+30€)–8×(10€+15€) falls to $500€-\left(\sum\boldsymbol{w}_i+\sum\boldsymbol{m}_j\right)$ =220€, $i,j=\overline{1,5}$. We refer to the list $\mathrm{D}\boldsymbol{x}=\langle\sigma\rangle$ as $\langle\sigma\rangle=R\div X$, or $X=R\div\mathrm{D}\boldsymbol{x}$; cf. Table 4-6 $\mathcal{W},\mathcal{M}\,\&\,X$. The list of matching pairs $\mathrm{D}\boldsymbol{x}$ is also the "complement list" $\overline{\mathrm{D}}\boldsymbol{x}$ of possible unmatched pairs in the sub-block X to R. Further removal of pairs $\alpha,\ldots$ from X will be denoted by $X\div\{\alpha\}$.

Based on the information provided, the matching that would best represent the common interests of all clients and agency staff is one that maximizes the least compensation sum, while maintaining the acceptable risk of incompatibility. What should be the matching that will represent the common interests of all clients and staff employees?

3. CONCEPT OF A QUASI-CORE—THE KERNELS

In coalition game theory, imputations refer to allocations of rewards that satisfy certain conditions, such as individual rationality, meaning that each player gets at least as much as they could have obtained on their own. However, marketing cannot be seen as game in traditional sense with a well-defined set of rules and a characteristic function. The concept of marketing presented so far as a game was just a framework for thinking in various directions at the marketing platform.

In view of "monotone system" (Mullat, 1971-1995) exactly as in Shapley's convex games, the basic requirement of our model validity emerges from an inequality of monotonicity $\pi(\alpha, X \div \{\sigma\}) \leq \pi(\alpha, X)$. This means that, by eliminating an element/match σ from X, the utilities (risks) on the rest will decline or remain the same. In particular, a class of monotone systems is called **p**-monotone (Kuznetsov et al, 1982, 1985, [17-18]), where the ordering $\langle \pi(\alpha, X) \rangle$ on each subset X of utilities follows the initial ordering $\langle \pi(\alpha, R) \rangle$ on the table R. The decline of the utilities on **p**-monotone system does not change the ordering of utilities on any subset X. Greedy type (serialization) technique on **p**-monotone system might be effective. Behind a **p**-monotone system lays the fact that an application of Greedy framework can accommodate the structure of all subsets $X \subset R$. For various reasons, many will probably argue that **p**-monotone systems are rather simplistic and cannot be compared with the serialization method. However, many economists, including Narens and Luce (1983, [19]), certainly, will point out that subsets X of **p**-monotone systems *perform* on interpersonally compatible scales.

An inequality $F(X_1 \cup X_2) \geq \min\langle F(X_1), F(X_2) \rangle$ holds for real valued set function $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$, referred to as quasi-convexity (Malishevski, 1998, [20]). We observed monotone systems here, which we consider important to distinguish. The system is non-quasi-convex when there are two sub-blocks X_1, X_2 contradicting the last inequality. We consider such systems as non-quasi-convex.

The order of incompatibility risks in marketing games may not be preserved within an arbitrary sub-block X. In these systems, the initial risks order $\langle R = r_{i,j} \rangle$ may not necessarily be true for some order on $\langle X \rangle = \|\pi(\alpha, X)\|$. Unlike $\langle R = r_{i,j} \rangle$, as agency staff employees search for an client for a marketing, and vice versa, the order of risks on $\langle X \rangle = \|\pi(\alpha, X)\|$ can be opposite to the order on $\langle R = r_{i,j} \rangle$ for some pairwise pairs α and β of participants, i.e. as $\pi(\alpha, R) > \pi(\beta, R)$, but $\pi(\alpha, X) \leq \pi(\beta, X)$ and the like. In that case, the ordering of two partners' mutual risks can turn "upside down" while the risks scale is dynamically narrowed down compared to the original ordering $\langle R \rangle$. This means that the scale of mutual risks is not necessarily interpersonally compatible. The interpersonal incompatibility of the risk scale in the marketing environment is significantly different, leading to difficulties in finding a solution using the Greedy framework and the incremental chain algorithm. This difference became apparent when the monotone system was found to be non-quasi-convex, making it impossible to find a solution using our traditional method (Mullat, 1971). Understanding the essence of the problem is essential before delving into the formal intricacies of the issue.

Definition 1 We call a sub-block $\mathcal{K} \in \arg\max_{X \subseteq \mathcal{P}} F(X)$ by a kernel sub-block; $\{\mathcal{K}\}$ is the set of all kernels.

Recalling the main properties of a chain of increments (a sequence of elements of a monotone system) it is possible to arrange the partners $\alpha \in \mathcal{P}$, i.e., the matchings $\alpha \in \mathcal{P}$ of agents by a Greedy type incremental sequence $\overline{\alpha} = \langle \alpha_1, ..., \alpha_k \rangle$, time slices $k = \overline{1, |\mathcal{P}|}$. The sequence $\overline{\alpha}$ follows the lowest risk ordering in each period k corresponding to sequence of sub-blocks $\langle H_k \rangle$, $H_1 = R$, $H_{k+1} \leftarrow H_k \div \{\alpha_k\}$, $\alpha_k = \arg\min_{\alpha \in H_k} \pi(\alpha, H_k)$. One of the properties of the incremental sequence (cf. *defining*, Mullat, 1971a) is that $F(H_k)$ is single-peaked. This means that within a peaked sub-block Γ_p for some time slice $k = p$ there does not exist a proper sub-block X' on which the function $F(X')$ would reach a greater value than on Γ_p, i.e., the inequality $F(X') > F(\Gamma_p)$ does not take place. Therefore, under the contrary assumption that such a set X' exists, X' must have a non-empty intersection with the sequence $\overline{\alpha}$ with some α_t in previous time slice; α_t will presumably be at the leftmost position $t < p$ in $\overline{\alpha}$ (or one of those $\overline{\alpha}$ entries α_t in X' that will appear as $\overline{\alpha}$ is constructed). However, complementing the pairs in X' with all those pairs that do not belong to X', so that starting from some t both X' and Γ_p lie entirely in the sequence $\overline{\alpha}$, we do not arrive to contradiction as expected while constructing $\overline{\alpha}$. Otherwise the sequence $\overline{\alpha}$ could potentially be used for finding the largest kernel $\mathcal{K}'$. The reason is, that incremental constructing the sequence $\overline{\alpha}$ is not an exclusion of matchings $\alpha_k \in H_k$, given that the participant $\alpha_k = [i, j]$ is about to match but rather an exclusion of all adjacent partners α in $[i,*]'$-s row and $[*, j]'$-s column. We denote this exclusion or dynamically reassessing of rows and columns by $H_{k+1} \leftarrow H_k \div \{\alpha_k\}$ and by $D_{k+1} \leftarrow D_k + \{\alpha_k\}$.

In conclusion, we note once again that, despite the preservation of the properties of a monotone system, the Greedy algorithm constituting Mullat's defining sequence $\overline{\alpha}$, the sequence cannot guarantee the extraction of the supposedly largest kernel $\mathcal{K}'$, especially in the form given by Kempner et al (2008, [21]). Thus, we need to employ special tools for finding the solution. To move further in this direction, we are ready to formulate some propositions for finding kernels $\mathcal{K}$ by branch and bound algorithm types.

The next argument will require a modified variant of imputation (Owen, 1982, [22]). We define an imputation as the outcome connected to the marketing game. More specifically, the outcome is given as a $|\mathcal{P}|$-vector (a list) of payoffs to all unmatched participants who make up the sub-block X, and matched participants as partners in pairs $\alpha = [\mathrm{L}_{\alpha}/\text{client}, \mathrm{E}_{\alpha}/\text{employee}] \notin \mathrm{X}$ representing the list $\mathrm{D}\boldsymbol{x}$. In case the game ends prematurely, at the request of the agency or the clients themselves, for all in $\mathrm{D}\boldsymbol{x}$ who have found a partner a reward $\mathrm{F(X)}$ will be paid; $\mathrm{F(X)} = \min \mathrm{r}_{\mathrm{i,j}}$ among cells $\alpha = [\mathrm{i,j}] \in \mathrm{X}$, cf. Table 3 and Table 6. For everyone who has not yet found a partner, under the current rules, they will receive $\frac{1}{2}\mathrm{F(X)}$. The concept of outcome (payoffs) in this form is not generally accepted as a form of imputation of a multi-persons game, since the amount that all participants can now claim is not fixed, but will be dynamically re-evaluated. Thus, it is likely that participants will fail to reach an understanding, and will claim payoffs obtaining less than entrance fees $(\mathrm{n}+\mathrm{m})\cdot 50\,€$ of the cashier. However, the cashier balance, in contrast, when participants will claim more than entrance fees, is also conceivable.

Any sub-block X induces a $|\mathcal{P}|$-vector $\boldsymbol{x} = \langle \boldsymbol{x}_{\sigma} \rangle$ as an outcome $\boldsymbol{x}$ may be organized in a sequence of payoffs $\langle \boldsymbol{w}_{\sigma}, \boldsymbol{m}_{\sigma} \rangle$. Further, we follow the rule that capital letters represent sub-blocks $\mathrm{X}, \mathrm{Y}, ..., \mathcal{K}, \mathcal{N}, ...$ while lowercase letters $\boldsymbol{x}, \boldsymbol{y}, ..., \boldsymbol{k}, \boldsymbol{n}...$ represent outcomes induced by these sub-blocks.

$$\boldsymbol{x}_{\sigma} = \begin{cases} \boldsymbol{w}_{\sigma}, \boldsymbol{m}_{\sigma} = 1 + \mathrm{F(X)} \text{ if } \sigma \in \mathrm{D}\boldsymbol{x}, \\ \boldsymbol{w}_{\sigma}, \boldsymbol{m}_{\sigma} = 1 + \frac{1}{2}\mathrm{F(X)} \text{ if } \sigma \notin \mathrm{D}\boldsymbol{x}. \end{cases}$$

The vector $\boldsymbol{x}$ designates an imputation in the terminology of many persons´ games, **1** stands for goodies:

$$\sum_{\sigma \in \mathcal{P}} \boldsymbol{x}_{\sigma} = \mathrm{F(X)} \cdot \left[|\mathrm{D}\boldsymbol{x}| + \tfrac{1}{2}\left(|\mathcal{P}| - |\mathrm{D}\boldsymbol{x}|\right) \right] + |\mathcal{P}|.$$

This definition of the partial matchings $\mathrm{D}\boldsymbol{x} \subseteq \mathcal{P}$ is used later, adapting the concept of the quasi-core for the purpose of the marketing game. We say that an arbitrary sub-block X induces an outcome $\boldsymbol{x}$. Computed and prescribed by sub-block X, the components of $\boldsymbol{x}$ consist of two distinct values $1 + \mathrm{F(X)}$ and $1 + \frac{1}{2}\mathrm{F(X)}$. Participants $\sigma \in \mathrm{X}$ could not sign a deal, while participants $\sigma \in \mathrm{D}\boldsymbol{x}$ were able to match. We will also use the notation $\overline{\mathrm{X}} \equiv \mathrm{D}\boldsymbol{x}$ emphasizing a mixture for marketing matchings $\mathrm{D}\boldsymbol{x}$.

Before moving on, let's try to justify our mixed notation $\overline{X}$. Although the cells $\alpha \notin X$, whereas α is located in the compliment $\overline{X}$ of X to R, the $D\boldsymbol{x}$ uniquely defines both those $D\boldsymbol{x}$ among participants $\mathscr{P}$ who signed deals, and those $X = R \div D\boldsymbol{x}$ who did not; the cells in $\overline{X}$ does not specifically indicate matched participants. In contrast, using the notation $D\boldsymbol{x}$, we indicate participants in $D\boldsymbol{x}$ who are matched, whereas $\sigma \in D\boldsymbol{x}$ also indicates an individual decision how to match. More specifically, this annotation represents all agency staff employees and all clients in $D\boldsymbol{x}$ like standing in line facing each other at the marketing platform. However, any agreement or matching among participants belonging to $D\boldsymbol{x}$, or whatever matches are formed in $D\boldsymbol{x}$, does not change the payoffs $\boldsymbol{x}_\sigma$ valid for the outcome $\boldsymbol{x}$. In other words, each particular matching $D\boldsymbol{x}$ induces the same outcome $\boldsymbol{x}$. Decisions in $D\boldsymbol{x}$ with respect to how to match provide an example of individual rationality, while the matching $D\boldsymbol{x}$ formation, as a whole, is an example of collective rationality. Therefore, in accordance with payoffs $\boldsymbol{x}$, the notation $D\boldsymbol{x}$ subsumes two different types of rationality—the individual and the collective rationality. In that case, the outcome $\boldsymbol{x}$ accompanying $D\boldsymbol{x}$ represents the result of a partial matching of participants $\mathscr{P}$. Propositions below somehow bind the individual rationality with the collective rationality.

The feasibility issue of induced sub-blocks $X \subset R$ is considered not only in the context of the blocks themselves, but also in the context of the totality $2^{\mathscr{P}}$ of matchings $D \in 2^{\mathscr{P}}$ in relation to special sets of matchings $\mathscr{F} \subset 2^{\mathscr{P}}$. The matching chain $\langle \alpha_k \rangle$ adding participants period-wise in the period k, starting with the empty set $\varnothing$, can, in principal, access any matching $D \in \mathscr{F}$, by removing the participants starting with the grand ordering $\mathscr{P}$—so called upwards or downwards accessibility.

Definition 2 *Given matching* $D \subseteq \mathscr{P}$*, where* $\mathscr{P}$ *is the Grand Coalition; we call the collection of pairs* $C(X) = \{\arg\min_{\alpha \in X} \pi(\alpha, X)\}$ *naming* $C(X)$ *as best latent participants, which can be matched with a minimum risk of mutual incompatibility in the matching* D.

Consider the formation of the chain $D_{k+1} \leftarrow D_k + \{\alpha_k\}$ of matchings D_k generated during in the periods $k = \overline{1, n}$. Let $X_1 = R$, $X_k = R \div D_k$, where D_k are participants trying to match; by Definition 2, these $C(X_k)$ are participants with the lowest risk of mutual incompatibility among participants D_k that do not yet matched in previous periods $k < k+1$, $D_{n+1} = \varnothing$. In the time slices $D_{k+1} = D_k \div \{\alpha_k\}$ the matching is arranged after the rows and columns, indicated by the matching or partners α_k, which have been removed from W, M and R. Mutual incompatibility risks $R = \|r_{i,j}\|$ have been recalculated accordingly.

Definition 3 *Given the sequence* $\langle \alpha_1,...,\alpha_k \rangle$ *of matched participants, where* $X_1 = R$, $X_{k+1} = X_k \div \{\alpha_k\}$, *we say that matching* $D\boldsymbol{x} \equiv \overline{X} \equiv R \div X$ *of matched (as well as* X *of not yet matched) participants is feasible, when the chain* $\langle X_1,...,X_{k+1} = X \rangle$ *complies with the rational succession* $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$. *We call the outcome* $\boldsymbol{x}$, *induced by sequence* $\langle \alpha_1,...,\alpha_k \rangle$, *a feasible payoff, or a feasible outcome.*

Proposition 1 *The succession rationality necessarily emerges from the condition that, under formation of the matching* D_k *partners in* α_k *does not decrease the payoffs of participants* $\langle \alpha_1,...\alpha_{k-1} \rangle$ *formed in previous periods.*

The accessibility or feasibility of matching $D\boldsymbol{x}$ formation offers a reinforcing interpretation. Indeed, the feasibility of matching $D\boldsymbol{x}$ means that the matching can be formed by bringing into it a positive increment of rankings to all participants $\mathcal{P}$, or by improving the position of existing participants having already formed the matching when new participants enter the matching in subsequent periods. We argue that in the subsequent periods, matching can be extended via hereditary-rational choice. In the addendum, we outline the heredity rationality in the form suitable for visualization.

The proposition states that, in matches, the individual decisions are also rational in a collective sense only when all participants in $D\boldsymbol{x}$ individually find a suitable partner. We can use different techniques to meet the individual and collective rationality by matching all participants only in $D\boldsymbol{x}$, which is akin to the stable marriage procedure (ibid [1], Gale & Shapley). In contrast, the algorithm below provides an optimal outcome/payoff accompanied by partial matching only—i.e., only matching some of participants in $\mathcal{P}$ as participants of $D\boldsymbol{x}$; once again, this is in line with the Greedy type matching framework. At last, we are ready to focus on our main concept.

Proposition 2 *The set* $\{\mathcal{N}\}$ *of kernels in the marketing game arranges feasible* matchings $\{D\boldsymbol{n}\}$. *Any outcome* $\boldsymbol{n}$ *induced by a kernel* $\mathcal{N} \in \{\mathcal{N}\}$ *is feasible.*

Definition 4 *Given a pair of outcomes* $\boldsymbol{x}$ *and* $\boldsymbol{y}$, *induced by sub-blocks* X *and* Y, *an outcome* y *dominates the outcome* $\boldsymbol{x}$ *by* $\boldsymbol{S}$, $\boldsymbol{x} \prec_S \boldsymbol{y}$:

(i) $\exists \boldsymbol{S} \subseteq X \cap Y \big| \forall \sigma \in \boldsymbol{S} \rightarrow \boldsymbol{x}_\sigma < \boldsymbol{y}_\sigma$, (ii) the outcome $\boldsymbol{y}$ is feasible.

Condition (i) states that participants/partners $\sigma \in \boldsymbol{S}$ receiving payoffs $\boldsymbol{x}_\sigma$ can break the initial matching and instead of merging into $D\boldsymbol{x} + \sigma$ and establish new matches will try to unite into $D\boldsymbol{y} + \sigma$. This means that, some partners in X, i.e., not yet matched participants in $\boldsymbol{S}$, can find suitable partners amid participants in $\boldsymbol{S}$, so that their compensations may be higher than their rewards in $\boldsymbol{x}$. Thus, by receiving $\boldsymbol{y}_\sigma$ instead of $\boldsymbol{x}_\sigma$ the participants belongings to $\boldsymbol{S}$ are guaranteed to improve their positions. This interpretation of the condition (ii) is obvious. Thus, the relation $\boldsymbol{x} \prec_S \boldsymbol{y}$ indicates that participants in $\boldsymbol{S}$ can cause a split (bifurcation) of $\mathrm{D}\boldsymbol{x}$, or are likely to undermine the outcome $\boldsymbol{x}$.

Definition 5 *The proper kernel $\mathcal{N} \in \{\mathcal{K}\}$ minimal by inclusion, or what is the same: a proper $\mathrm{D}\boldsymbol{n}$, maximal by super-matchings′ induced by $\mathcal{N}$, is called a core kernel or matching.*

Proposition 3 *The set $\{\boldsymbol{n}\}$ of outcomes, induced by core kernels in $\{\mathcal{N}\}$, arranges a quasi-core of the marketing game. Outcomes in $\{\boldsymbol{n}\}$ are non-dominant upon each other i.e., $\boldsymbol{n} \prec_S \boldsymbol{n}'$, or $\boldsymbol{n} \succ_S \boldsymbol{n}'$ are false for any $\boldsymbol{S} \subset \mathcal{N} \cap \mathcal{N}'$.* Thus, the quasi-core is internally stable.

The proposition above indicates that the concept of internal stability is based on "pair comparisons" (binary relation) of outcomes. The traditional solution of marketing games recognizes a more challenging stability, known as *NM* solution, which, in addition to the internal stability, demands external stability. External stability ensures that any outcome x of the game outside *NM*-solution cannot be realized because there is an outcome $\boldsymbol{n} \in \{\mathcal{N}\}$, which is not worse for all, but it is necessarily better for some participants in $\mathrm{D}\boldsymbol{x}$. Therefore, most likely, only the outcomes $\boldsymbol{n}$ that belong to *NM*-solution might be realized. The disadvantage of the marketing scenario is that it is impossible to specify how this can happen. In contrast, we can define how the dynamic or multi-stage reassessment of one matching to another takes place, namely, only along feasible sequence of matchings of partners. However, it may happen that for some matchings $\mathrm{D}\boldsymbol{x}$ outside the quasi-core $\{\mathcal{N}\}$, "feasible sequence" may come to deadlock unable to reach any better outcome that $\boldsymbol{n}$, whereby starting at $\mathrm{D}\boldsymbol{x}$ the quasi-core is feasibly unreachable. This is a significant difference with respect to the traditional *NM*-solution.

4. CONCLUSIONS

By using mismatch or incompatibility indicators as metrics, we can identify cases where a partial matching may be more beneficial than a complete matching. For example, if a staff member has a high level of expertise in a certain area, but may not be a perfect matching for a particular client, it may be better to assign them to that client anyway, rather than risking a less experienced staff member who is a better matching. By reassessing
these metrics throughout the marketing process, we suppose that participants can see, i.e., to reflect all the consequences of their partial matchings as well as actions of their partners to achieve better results overall. This approach may result in a higher total reward than a complete or grand matching, which may not always be feasible or desirable in practice.

The marketing game dynamically develops in time. The scenario is described by multi-stage decision process from current reflection k to the next reflection $k+1$ in the form of a dynamic reassessment of indicators about the willingness to take the risk of entering into incompatible agreements. The objection being raised is that the model presented is not a strict strategic interaction, but rather a "game" in the colloquial sense. On the contrary, it is noted that at each reflection, agents have multiple options and time to consider their moves, including the option to leave the game and receive a payoff or to continue in the hope of obtaining a better outcome. This allows for a more flexible and nuanced approach to cooperative game theory, which is more in line with mathematical standards. The model uses scalar optimization based on the Rawlsian principle of "maximum welfare of the worst-off". In summary, the design of the marketing game should prioritize the promotion of services to clients and benefits to staff, while also providing an engaging and challenging experience for players.

The uniqueness of the marketing game lies in dynamic reassessment of clients and agency staff employees on each other's risks to make deals. As a result, along with the individual and pair rankings, the collective ranking is also subject to reassessment. Indeed, the agreements or matchings indicate the collective action that each agent (clients or staff employees) must take to prepare a suitable deal at each reflection k of the game. This situation is manifested by the construction of an appropriate sequence of risks that increase at the starting periods $1 < ..k .. <<$ and then dynamically decrease in game closing periods $1 << .. k < ..$ The sequence of risks of incomparability of matchings finally, albeit in the most unfavorable case, converges to a "single point" at the end of the game. The reassessment of risks has a monotonic character, which made it possible to build a game based on the so-called Monotone system (MS).

One disadvantage of the MS is the challenge in aligning the results of the analysis with a realistic interpretation. The quasi-core extraction process may require additional adjustments for proper interpretation. However, the idea of using mismatch or incompatibility indicators as metrics can help to identify latent issues before they arise, and allow marketing agency staff employees to proactively manages the situation. By measuring compatibility between a marketing agency employee and a client, the agency can make more informed decisions about who to assign to each client, ultimately leading to improved customer satisfaction and reduced marketing volatility and fuzziness. It is important to note that $r_{i,j}$ metrics could be used in conjunction with other factors, such as the time frame as explained in Osborne and Rubinstein (2020, [23]).

The question being raised is why a partial matching (in this case, a pairwise matchings of agents) is preferable to a complete matching. In constructing "Greedy-type" the multi-stage time sequences becomes a single-peaked. As a result, a partial matching in the form of quasi-core imputations is more preferable or efficient than a complete matching.

The concept of the core in cooperative game theory refers to the sets of feasible payoffs that can be achieved by the players through cooperation. Finding the exact payoffs, associated with the core, is difficult problem meaning that it may not be solvable using current computing power. The problem becomes unclear also because, among other things, it is not known whether the core is empty. The existence of non-empty payoff sets, similar to the core, called quasi-cores, is guaranteed in marketing game. A quasi-core is defined as a stable sets determined by the marginal values of supermodular utility functions, in accordance with Rawls' second principle of justice. These sets can be identified using a version of the P-NP problem that uses the branch and bound heuristic, which is an optimization algorithm that combines a systematic search in the solution space with checking upper and lower bounds of the remaining subtasks. The heuristic can be visualized using spreadsheets such as Microsoft Excel where an optimization problem can be modeled and solved with a combination of formulas and algorithms. The branch and bound heuristic can give approximate solutions in a relatively efficient way, allowing a rough estimate of the quasi-core.

The quasi-core concept in marketing game refers to a fundamental idea or principal that guides marketing activities. It can be applied to marketing to evaluate the stability of marketing comparisons and determine whether a given marketing strategy is feasible. The stability of the marketing depends on how well it is able to account for externalities, such as the actions of competitors, changes in consumer preferences, and the impact of technology. By analyzing the stability of matchings in the context of the quasi-core, marketing professionals can gain insights into the likelihood that their strategy will be successful and be able to make informed decisions about how to adjust their approach as necessary. In this sense, the quasi-core concept can be seen as a tool for promoting the long-term viability of marketing initiatives.

APPENDIX

A1. Addendum

To understand what is proposed below, the situation is such that the game can be viewed as a dynamic or multi-stage reassessment of rankings by reflections $.,k\,,k+1,.$, as a chain shrinking sub-blocks from $.,X_k$ to $X_{k+1},.$. When participants as pairs or partners α signed an agreement and reserved their services and products, the sub-blocks $X_k \supset X_{k+1}$ are reassessed or narrowed down. If among best matches $C(X_{k+1})$ in X_{k+1} there are matches from X_k, then, in this new situation X_{k+1}, these best pairs $C(X_k)$ from X_k should be present in their former role as the best choice, especially true for matchings in the quasi-core.

One circumstance must be kept in mind here. On the one hand, we are dealing with matchings, but on the other hand, the considered matchings are also a certain set of cells or sub-blocks X embedded into $n \times m$ tables in our marketing game, and therefore it is quite appropriate to consider matchings from the point of view of Boolean set theory, where the usual operations of inclusion, intersection of table cells as pairs of participants, etc. are allowed.

As the sub-block-formation chain X_k shrinks $\langle X_k \supset X_{k+1} \rangle$ the proposition below can be verified by least-risk $F(X_k) = \min_{\sigma \in X_k} \pi(\sigma, X_k)$ generating choices $C(X_k)$ as a list $\langle \alpha_k = \arg\min_{\sigma \in X_k} \pi(\sigma, X_k) \rangle$ of potential participants $\alpha_k \in X_k$ at risks levels $F(X_k)$. The list $C(X_k)$ represents matchings $\overline{\alpha} = \langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$ that participants $\overline{\alpha}$ decide to match. Partners $\sigma \in X_{k+1}$ now in the role some $\alpha_{k+1} = \sigma$ will try to realize their latent relations. While the chain X_k has been formed, due to the fact that all participants in $\overline{\alpha}$ no longer are available (reserved) for new matching, in the new reflection $k+1$, all eventual partners/cells in X_{k+1}, must reconsider to whom they prefer to match, as their favored σ. Based on the remarks above, the following can be stated.

Proposition 5. *In the marketing game, the participants of the game move from the best choice* $C(X_k)$ *on previous period of the game to the next best choice* $C(X_{k+1})$ *on succeeding period. If it turns out that in succeeding period* X_{k+1} *the old bests* $C(X_k)$ *are still present, i.e.,* $C(X_k) \cap X_{k+1} \neq \varnothing$ *, then* $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$*. These* $C(X_k)$ *"old best clients" will continue to be the best for marketing by the same staff employees of the agency, provided that the reward payments* $F(X_k)$ *will not increase:* $F(X_k) = F(X_{k+1})$.

The proposition somehow revises a rational mechanism of so-called heredity succession choice $C(X)$; Postulate 4, Chernoff (1954, [24]), condition α of Sen (1970, [25]), or fuzzy form [26], cf. Arrow Axiom (1959, [27]; cf. also Malishevski [20]. The proof may be explained in the basic terms. It is possible to reach an arbitrary sub-block X not yet matched participants by sequence $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$, $X_1 = R$, $X_{k+1} = X_k \div \{\alpha_k\}$ $X = X_{k+1}$, starting from the initial reflection R of the game, where nobody has been matched yet. The sequence will improve payoffs $\boldsymbol{x}_k$ on previous periods $\langle \alpha_1, ..., \alpha_k \rangle$ in accordance with non-decreasing values $F(X_k)$.

The statement of the proposition can be verified by observation of all priority tables and all matchings X that emerged from all $n \times m$ tables, when both n and m are small integers. For higher n and m values, it is NP-hard problem. Second, consider an arbitrary sub-block X of the $n \times m$ -game. While the anti-sub-block $\overline{X} \equiv D\boldsymbol{x}$ includes all participants signed a deal; all participants in X are still unmatched. We can thus always find partners $\alpha_1 \in \overline{X}$ such that $F(R) \le F(R \div \{\alpha_1\})$. Consider $(n-1)\times(m-1)$-game, which can be arranged from $n \times m$-game by declaring the partners signed a deal α_1 as blank agents, $\boldsymbol{i}_{\alpha_1}, \boldsymbol{j}_{\alpha_1} \notin \mathcal{P}$.

By the induction scheme, there exists a sequence of matchings $\langle \alpha_1, \alpha_2 ..., \alpha_k \rangle$ with required quality of improving the payoffs X_k starting from $X_1 = R \div \{\alpha_1\}$. Restoring the blank attendees α_1 to the role of clients and agency staff employees in the $n \times m$-game, we can, in particular, ensure the required quality of the sequence $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$. The statement of the proposition is obviously the corollary of the claim above. However, ensured by its logic, the claim is a more general statement than the statement of the proposition. The first part of the statement is self-explanatory. The matching $\mathcal{N}$ stops being a proper subset among kernels $\{\mathcal{K}\}$ as soon as the payoff function $F(\mathcal{N})$ do not allow improving the outcome $\boldsymbol{n}$. The second part of the proposition is the same statement, worded differently. Nonetheless, we consider it necessary to provide complete proofs of all statements, since proofs are presented here only in a concise form.

A2. Finding the quasi-core

In general, algorithms like Greedy improve the solution dynamically through reassessment. However, in the case of the marketing game, this approach is complicated by the fact that local improvements may not necessarily lead to the best outcome or payoff for all agents. The best outcomes for all agents make up the quasi-core of the marketing game, and there may be numerous best compensations. Finding the core in the conventional sense is NP-hard because the number of operations increases exponentially with the number of participants. In the marketing scenario and other marketing games, there is a large family of subsets that make up the traditional basis of imputations. While it may be possible to find all payoff vectors induced by kernels, it is impractical to do so. Therefore, we suggest finding some admissible matchings belonging to the quasi-core and the payoffs induced by these matchings are sufficient.

This can be achieved by applying a strong payoff improvement procedure and several rolling procedures that do not worsen the position of the agents when forming the matching. In some situations, known as succession rationality, Definition 3, the strong improvement procedure cannot find anything. On the contrary, using rolling procedures, we can move forward in one of the promising directions to find payoffs that do not worsen the result. Experiments performed using our polynomial algorithm show that by using a combination of improvement procedures and rolling procedures both with a rational succession, it is possible to use a backtracking search strategy and find possible payoffs belonging to the quasi-core.

We use five procedures in total—one improvement procedure and four variants of rolling procedures. Combining these procedures, the algorithm below is given in a more general form. While we do not aim to explain in detail how to implement these five procedures, in relation to rational succession, it will be useful to explain beforehand some specifics of the procedures because a visual interaction is best way to implement the algorithm.

In the algorithm, we can distinguish two different situations that will determine in which direction to proceed. The first direction promises an improvement in case the attendees $\alpha \in X$ decides to match or sign a deal. We call the situation when $C(X \div \{\alpha\}) \cap C(X) = \varnothing$ as a latent improvement situation. Otherwise, when $C(X \div \{\alpha\}) \cap C(X) \neq \varnothing$, it is a latent rolling direction. Let $CH(X)$ be the set of rows $C(X)$, the horizontal routes in R Table 3 & 6, which contain the set $C(X)$. By analogy $CV(X)$ represents the vertical routes, the set of columns, $C(X) \subseteq CH(X) \times CV(X)$. To apply our strategy upon X, we distinguish four cases of four non-overlapping blocks in the mutual risk $R = \|r_{i,j}\|$ Table 3 & 6: $CH(X) \times CV(X)$; $CH(X) \times \overline{CV(X)}$; $\overline{CH(X)} \times CV(X)$; $\overline{CH(X)} \times \overline{CV(X)}$.

Proposition 4 *An improvement in payoffs for all participants in the marketing game may occur only when partners $\alpha \in X$ comply with the latent improvement situation in relation to the sub-block X, the case of $C(X \div \{\alpha\}) \cap C(X) \neq \varnothing$. The attendees $\alpha \in X$ are otherwise in a latent rolling situation.*

The following algorithm represents a heuristic approach to finding payoffs $\boldsymbol{n}$ induced by kernels $\{\mathcal{N}\}$ of the marketing game. Recall that R is the notation for the table of mutual risks. Build the mutual risks Table 3 & 6, $R = W + M$—a simple operation in Excel spreadsheet.

Input Set $k \leftarrow 1$, $X \leftarrow R$ in the role of not yet matched participants, i.e., as agents available for latent matching. In contrast to the set X, allocate indicating by $D\boldsymbol{x} \leftarrow \varnothing$ the initial status of matched participants.

Do: **S**, Find a match $\alpha_k \in CH(X) \times CV(X)$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) < F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else *Track Back.*

Rolling: **D**, Find a match $\alpha_k \in CH(X) \times CV(X)$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else *Track Back.*

JumpF **F**, Find a match $\alpha_k \in CH(X) \times \overline{CV(X)}$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else *Track Back.*

JumpG **G**, Find a match $\alpha_k \in \overline{CH(X)} \times CV(X)$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else *Track Back.*

JumpH **H**, Find a match $\alpha_k \in \overline{CH(X)} \times \overline{CV(X)}$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else *Track Back.*

Loop Until no participants can be found in accordance with macros **S, D, F, G** and **H.**

Output The set $D\boldsymbol{x}$ forms $D\boldsymbol{x} = \langle \alpha_1, ..., \alpha_k \rangle$. The row-column removal of $D\boldsymbol{x}$ from R, $\mathcal{N} = R \div D\boldsymbol{x}$, represent the technical framework of the game while the payoff $\boldsymbol{n}$ induced by $\mathcal{N}$ belongs to the quasi-core.

In closing, it is worth noting that a technically minded reader would likely observe that matchings X_k are of two types. The first case is $X \leftarrow X \div \{\alpha_k\}$ operation when the mismatch compensation for bad luck increases, i.e., $F(X_k) < F(X_k \div \{\alpha_k\})$. The second case occurs when rolling along the compensation $F(X_k) = F(X_k \div \{\alpha_k\})$. In general, independently of the first or the second type, there are, as said, five different directions in which a move ahead can proceed. In fact, this poses a question—in which sequence of participants α_t should be selected in order to facilitate the generation of the *sequence* $D\boldsymbol{x} = \langle \alpha_1, ..., \alpha_k \rangle$ of matchings? We solved the problem for marketing games underpinning our solution by backtracking. It is often clear in which direction to move ahead by selecting improvements, i.e., either a strict improvement by **s)** or rolling procedures though **d)**, **f)**, **g)** or **h)**. However, a full explanation of backtracking is out of the scope of our current investigation. Thus, for more details, one may refer to similar techniques, which effectively solve the problem (Dumbadze, 1989, [28]).

A3. Conventional stability

In order to demonstrate the shortcomings, at least in one particular case, of using traditional game theory concepts such as the core, below we use a mixture of common game theory terms and try to show that the standard core does indeed give a rather poor solution as the core consists of a single imputation in the form of complete or grand matching. This suggests that alternative approaches may be required to solve the marketing game effectively.

The marketing game arrangement is expanded to a more general case. There are $n+m$ participants n of which are clients $\langle 1, ..., i, ..., n \rangle$, and m are agency staff employees $\langle 1, ..., j, ..., m \rangle$. Some of the participants expressed their willingness to participate in the game and have revealed their rankings. Those who refused are referred to as *blanks*, while others who agreed to play the game will be arranged by default into the Grand Matching $\mathcal{P}$, $|\mathcal{P}| \leq n+m$.

Indices i, j annotate the participants of the game. Participants in $\mathscr{P}$ are regarded as *players*, whereas partners $\alpha = [\mathrm{i}, \mathrm{j}]$... or $[\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha]$... are designated to as $\alpha, ..., \sigma$. This differentiation helps making notations short.

Marketing game focuses on the participants $\mathrm{D} \subseteq \mathscr{P}$ that are matched. Having formed their rankings, participants in D have the power and ability to assert their rankings. Participants in D can convince all those in $\overline{\mathrm{D}}$ who are not already in D to opt out of the game without a partner and thus be compensated. Given the tables W, M and R, the situation, in contrast to D, which lists matched pairs, i.e., those who made deals, can be represented as a sub-block $\mathrm{X} = \mathrm{R} \div \mathrm{D}$ consisting of rows and columns from $\overline{\mathrm{D}}$.

It is realistic to assume that enforcing the interests of the participants in D is not always possible. Regardless of their participation in D those in the $\mathrm{D}' \subset \mathrm{D}$, whose interests are affected (suppressed), will still be able to receive as much as they receive in D. Sometimes it is convenient for D' to exclude this opportunity, since it is better that the D' matching cannot be implemented simultaneously with D and be its direct competitor.

N.B. It should be emphasized here that the D matching are those participants who have signed deals, and the X sub-block are those who prefer to continue. Matching D and sub-block X characterize the game multi-stage situation achieved in period k, when the participants imitating each other actions must decide on the further course of the game, whether to move to reflection $\mathrm{k}+1$ or not. Each agent identified by the rows and columns in X receives 50% of the rewards of the agents in D in the event of the game is over. A realistic situation may occur when all participants in $\mathscr{P}$ are matched, $\mathrm{D} = \mathscr{P}$, or, in contrast, no one decides to match, $\mathrm{D} = \varnothing$ hereby after revealing their rankings, all might decide not to proceed with the game at all.

Among all matchings D, rational matchings are usually singled out. A participant, entering into the matching D, derives from the interaction in the matching a reward that satisfies $\alpha \in \mathrm{D}$. We assume that the rewards and compensations are strictly dependent on pairwise matchings in D, which in turn were caused by sub-block X. Using the matchings $\mathrm{D} \subseteq \mathscr{P}$, we can always construct a payoff $\boldsymbol{x}$ to all participants $\mathscr{P}$, i.e., we can quantify the positions of all participants. The inverse is also true. Given a payoff $\boldsymbol{x}$, it is easy to establish which participant belongs to the matching D and identify those belonging to block $\mathrm{X} = \mathrm{R} \div \mathrm{D}$. We label this fact also as $\mathrm{D}\boldsymbol{x}$. Recall that participants of the matching $\mathrm{D}\boldsymbol{x}$ receive a reward to match, which is equal to the double amount of the “mismatch” compensation. Thus moving to better positions, the list of participants $\mathrm{D}\boldsymbol{x}$ may provide an opportunity for some participants $\sigma \in \mathscr{P}$ to start, or initiate, new matches. We will soon see that, while the best positions induced by special sub-blocks $\mathscr{K}$, called the kernel block, have been reached, this movement will be impossible to realize. Our terminology is unconventional in this connection.

The concept of stability in matching games refers to the inability of agents to move to better positions by making pairwise comparisons. In the work "Cores of Convex games" by Shapley (1971, [39]) convex games were studied, which are games that have a non-empty core. The core is a convex set of end-points (imputations), representing the available payoffs to all agents in a multidimensional octahedron. The core stability in these games ensures that no agent has an incentive to move from their current position to a better one, leading to a stable solution. Below, despite the agents' asymmetry with respect to $\mathrm{D}\boldsymbol{x} = \mathrm{R} \div \mathrm{X}$, we focus on their payoffs driving their collective behavior as participants $\mathscr{P}$ to form the matching $\mathrm{D}\boldsymbol{x}$, $\mathrm{D}\boldsymbol{x} \subseteq \mathscr{P}$; $\overline{\mathrm{X}} \equiv \mathrm{D}\boldsymbol{x}$ is and anti-sub-block to X; $\overline{\mathrm{X}}$ designates deleted rows and columns.

In contrast to individual payoffs improving or worsening the positions of participants, when playing the marketing game, the total payment to the matching $\mathrm{D}\boldsymbol{x}$ as a whole is referred to the utility function $\boldsymbol{h}(\mathrm{X}) > 0$. In classical cooperative game theory, the payment $\boldsymbol{h}(\mathrm{X})$ to matching $\mathrm{D}\boldsymbol{x}$ is known with certainty, whereby the variance $\boldsymbol{h}(\mathrm{X}) - \boldsymbol{h}(\mathrm{X} \div \{\sigma\})$ provides a marginal utility $\pi(\sigma,\mathrm{X})$. Inequality $\pi(\alpha,\mathrm{X} \div \{\sigma\}) \le \pi(\alpha,\mathrm{X})$ of the scale of risks of incompatible agreements expresses a monotonic decrease (increase) in marginal utilities $\pi(\alpha,\mathrm{X})$ for $\alpha = [\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha] \in \mathrm{X}$. This monotonicity is equivalent to supermodularity $\boldsymbol{h}(\mathrm{X}_1) + \boldsymbol{h}(\mathrm{X}_2) \le \boldsymbol{h}(\mathrm{X}_1 \cup \mathrm{X}_2) + \boldsymbol{h}(\mathrm{X}_1 \cap \mathrm{X}_2)$, Nemhauser et al, 1978, [30]. Any utility function $\boldsymbol{h}(\mathrm{X})$, payments for which are built on a scale of risks of incompatible agreements, due to monotonicity, is supermodular. Supermodular functions have been used to solve many combinatorial problems (Petrov & Cherenin 1948, [31]; Emonds 1970, [32]; Bai & Bilmes, 2018, [33]). In general, a supermodular guarantee cannot be given.

Recall that we eliminated all rows and columns X in tables $\mathrm{W} = \|\mathrm{w}_{i,j}\|$, $\mathrm{M} = \|\mathrm{m}_{i,j}\|$ in line with $\overline{\mathrm{X}} \equiv \mathrm{D}\boldsymbol{x}$. Table $\|\mathrm{w}_{i,j}(\mathrm{X}) + \mathrm{m}_{i,j}(\mathrm{X})\|$ or $\|\pi(\alpha,\mathrm{X})\|$, where $\alpha = [\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha] \in \mathrm{X}$ imitates the dynamic outcome of dynamically reassessing rankings $\mathrm{w}_{i,j}$, $\mathrm{m}_{i,j}$ when some participants $\sigma \in \overline{\mathrm{X}}$ have been matched and signed a deal. Rankings $\mathrm{w}_{i,j}$ and $\mathrm{m}_{i,j}$ are consequently decreasing. Given in the form of utility function, e.g., the value $\boldsymbol{h}(\mathrm{X}) = \sum_{\alpha \in \mathrm{X}} \pi(\alpha,\mathrm{X})$ sets up the marketing game. An imputation for the game $\boldsymbol{h}(\mathrm{X})$ is defined by a $|\mathscr{P}|$-vector fulfilling two conditions: (i) $\sum_{\alpha \in \mathscr{P}} (\boldsymbol{w}_\alpha + \boldsymbol{m}_\alpha) = \boldsymbol{h}(\mathscr{P})$, (ii) individual rationality $\boldsymbol{w}_\alpha, \boldsymbol{m}_\alpha \ge \boldsymbol{h}(\{\alpha\})$, for all $\alpha \in \mathscr{P}$. Condition (ii) stems from repetitive use of monotonic inequality $\pi(\alpha,\mathrm{X} \div \{\sigma\}) \le \pi(\alpha,\mathrm{X})$.

A significant shortcoming of the canonical cooperative theory is related to its inability to define stable matchings (the core is empty) or consisting of only one—the grand matching. At first glance, this shortcoming seems inevitable. Indeed, the lower is the risk $\pi(\alpha, X)$ of incompatible matching $\alpha \in X$, the more reliable the matching $\alpha = [\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha] \in X$ will be. Let we set up as an exercise a popularity index u_i of client i among agency staff employees $D\boldsymbol{x}$ as $u_i(X) = \sum_{j \in X} m_{i,j}$; accordingly, the index u_j of an employee j popularity among clients will be given by $u_j(X) = \sum_{i \in X} w_{i,j}$. Let we intend to redistribute the payment $\boldsymbol{h}(\mathcal{P})$ of the complete matching $\mathcal{P}$ in proportion to the components of the vector $u(\mathcal{P}) = \langle u_i(\mathcal{P}), u_j(\mathcal{P}) \rangle$. Hereby we can prove, owing to monotonic inequality, that the payoffs in imputation $u(\mathcal{P})$ cannot be improved for any $\alpha \in \mathcal{P}$ inside any partial matching $D\boldsymbol{x} \subset \mathcal{P}$ induced by the sub-block X. Therefore, the game solution, among popularity indices, will be the only imputation $u(\mathcal{P})$—popularity indices core of the cooperative game consists of only one point $u(\mathcal{P})$. In other words, for matching all participants, any matching using any algorithm (in particular, ibid. Roth & Sotomayor) will be the best matching in terms of cooperative game using the only imputation $u(\mathcal{P})$.

A4. Visualization

Recall that, the input to the algorithm presented in the main body of the paper contains three tables (cf. Table 1-6): $W = \|w_{i,j}\|$—rankings table w_i where the client specify with the respect to the characteristics the agency staff employees should possess, in the form of permutations of numbers $\overline{1,n}$ in rows; $M = \|m_{i,j}\|$—visa versa, rankings m_j where staff employees specify the characteristics in the form of clients permutations of numbers $\overline{1,m}$ in columns; and $R = \|w_{i,j} + m_{i,j}\|$. These tables, and tabular information in general, are well suited for use in Excel spreadsheets that feature calculation, graphing tools, pivot tables, and a macro programming language called VBA—Visual Basic for Applications.

A spreadsheet http://datalaundering.com/download/marketings_game.xls (accessed December 23, 2021) was designed to visually represent our idea of finding the quasi-core $\overline{\alpha} = \langle \alpha_1, ..., \alpha_{12} \rangle$ of the marketing game, including the stable matchings that belong to the quasi-core. It was compiled by macro-activated rendering capabilities of Excel.

A5. Spreadsheet layout specification

Three tables are available: the Pink table W —client's rankings, the Blue M — agencies' rankings. The Yellow R —table consists of mutual risks $r_{i,j} = w_{i,j} + m_{i,j}$ of matchings incompatibility. The rows and columns, which represent those who ceased the game, will be highlighted with a gray shadow. According to this representation, the yellow sub-block X in R will represent all potential or new opportunities of the matching $\sigma = [i_\sigma, j_\sigma] \in X$. The global $F(X) = \min \to r_{\sigma \in X}$ occupy the cell in the lower right corner of the table R. The line on the right to X shows the minimum risk in the row $i_\sigma \in X$, and the horizontal line below X shows the minimum risk in the column $j_\sigma \in X$. The green cells in the yellow sub-block X visualize the choice operator $C(X) = \{\arg\min \to r_{\sigma \in X}\}$. The cells **[V24:AO25]** and **[V26:AO26]** contain the sequence $\overline{\alpha} = ..., X_k \supset X_{k+1}, ...$ of the game generated in periods $..., k, k+1, ...$ together with the risks of matching associated by the sequence. The agents' balance of payoffs occupies the cells [**V31:AO32**]. Some cells reflecting the *state of finances* of cashier are located below, in the cells [**AP34:AP44**]. Cells in row-**1** and column-**A** contain the participants' labels. We use these labels in all macros.

A6. Extracting the quasi-core of the game

We came closer to the goal of our visualization, where we visually demonstrate the main features of the theoretical model of the game by example. Generating the matching sequence, which is performed in a period-wise fashion, constitutes the framework of the theory. In each period, to the right side of the sequence generated in the preceding periods, we add partners found by one of the macros Case**S**, Case**D**, Case**G** and Case**H**, i.e., partners that has decided to match. This process is repeated until the marketing risks of incompatibility matching reach the level 6. When using these macros one can easily verify that, risks initially increase, and then decline towards the end in case we proceed further with these macros. This marketing $\cap$-peakedness is a consequence of the mutual risks of matching monotonicity $\pi(\alpha, H \div \{\sigma\}) \le \pi(\alpha, H)$. Indeed, recall that matching' levels are recalculated after each matching. With the proviso of recommendations in our heuristic algorithm, see above, due to the recalculation, the priority scales will "shrink" or "pack together", as only not yet matched participants remain. The sequence $\overline{\alpha}$ can be generated by macros: Case**S**, Case**D**..., Case**H**. The output will occupy the cells **[V24:O28]**. The initial reflection of the table can be restored with macros: Ctrl+**o**, Ctrl+**b** and Ctrl+**l**. As an example of these macros, we prepared the result in cells **[B51:L52]**. Just copy the contents of these cells into **[V24:F25]** and then use the Ctrl+n macro, which renders the core of the 11 matches of the game.

Table 7 *Attendees' belonging to the kernel*

Matches W_i / M_j	19	10	1	6	4	11	17	9	5	2	15
	5	9	10	17	15	6	13	11	7	14	2
Greedy risks' sequence	3	3	4	5	6	6	6	6	6	6	6

Table 8 ***Payoffs' imputation induced by the Kernel***

Agent/Moderators Id Nr., 1,..,10	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
w -payoffs	70 €	40 €	40 €	70 €	70 €	70 €	40 €	40 €	40 €	70 €
m -payoffs	70 €	40 €	70 €	70 €	70 €	70 €	40 €	40 €	70 €	70 €
Agent/Moderators Id Nr., 11,..,20	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
w -payoffs	70 €	40 €	70 €	40 €	70 €	40 €	40 €	70 €	70 €	70 €
m -payoffs	40 €	70 €	70 €	70 €	70 €	40 €	40 €	40 €	40 €	40 €

Let us look at Table 7, where only 11 matches are accomplished, i.e., all columns to right starting at from the match [19,5] till [15,2] visualize the outcome ***n*** of our marketing game. Table 7 marks those participants who decided to match, while all the rest but on this particular list are not yet taken their decisions or have been, perhaps, unlucky to find a partner.

Table 8 will note the payoffs, that is, the imputation induced by the kernel matching—the amount of payments in the form of rewards or compensations for bad luck to all 40 participants—20 clients and 20 agencies. Payoffs of 40€ and 70€ correspond to what the kernel makes up in cash. The result is a total amount of 2000€ received by the cashier in the form of participation fees minus 2260€ as payoffs, i.e., –260€ not in favor of the cashier.

We can continue creating the sequence of matchings with macros using m**A**tch [ctrl + a], pointing to the cell in the top box: pink on the left (or yellow on the right), until all participants have been matched. Please note this, starting with pair **No.12**; we can no longer use the macros of our heuristic algorithm. There are no participants with increasing payoff compensations 1-11, which represent the maximum point—a payoffs ***n*** of the game.

In the Table 9-10 below, the Matching Sequence consists of $k = \overline{1,20}$ time slices or periods; we labeled attendees $[i, j]$ using notation α_k. Together with levels of mutual risks in row 3, the pink and blue rows correspond to the sequence $\overline{\alpha} = \langle \alpha_k \rangle$ of matchings. Compensations and rewards for marketing are not payable at all, and only the costs of goodies (each worth 10€) occupy similarly pink and blue rows. For match #3, the participants risk jumps from 4 to 5, and for match number 4 also increase from 5 to 6. Note that due to the risks single $\cap$-peakedness, the lowest risk levels first for match #3 increase starting from 4, and after the level 6, starting from match #12, it begins to decrease to 0.

Table 9 *Total Match*

Match No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Matches W_i/M_j	19	10	1	6	4	11	17	9	5	2
	5	9	10	17	15	6	13	11	7	14
Greedy Risks	3	3	4	5	6	6	6	6	6	6
w -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
m -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

Table 10

Match No.	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Matches W_i/M_j	15	18	20	7	13	16	8	14	3	12
	2	1	4	12	20	18	19	3	16	8
Greedy Risks	6	5	5	4	3	3	3	3	2	0
w -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
m -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

The list of macros; CH —cells in horizontal, CV —cells in vertical direction.

- **CaseS.** Ctrl+**s,** Trying to move by improvement along the block $\mathrm{CH(X)} \times \mathrm{CV(X)}$ of cells $\sigma = [i, j]$ by "<" operator in order to find a new matching at the strictly higher level.
- **CaseD.** Ctrl+**d,** Trying to move while rolling along the block $\mathrm{CH(X)} \times \mathrm{CV(X)}$ of cells $\sigma = [i, j]$ by "<=" operator in order to find a new matching at the same or higher level.
- **CaseF.** Ctrl+**f,** Trying to move while rolling along the block $\mathrm{CH(X)} \times \overline{\mathrm{CV(X)}}$ of cells $[i, j]$ by "<=" operator in order to find a new matching at the same or higher level.
- **CaseG.** Ctrl+**g**, Trying to move while rolling along the block $\overline{\mathrm{CH(X)}} \times \mathrm{CV(X)}$ of cells $\sigma = [i, j]$ by "<=" operator in order to find a new matching at the same or higher level.
- **CaseH.** Ctrl+**h**, Trying to move while rolling along the block $\overline{\mathrm{CH(X)}} \times \overline{\mathrm{CV(X)}}$ of cells $\sigma = [i, j]$ by "<=" operator in order to find a new matching at the same or higher level.

Functional test. The spreadsheet users are invited first to perform a functional test, in order to become familiar with the effects of **ctrl-keys** attached to different macros. Calculations in Excel can be performed in two modes, ***automatic*** and ***manual***. However, it is advisable to choose properties and set the calculus in the manual mode, as this significantly speeds up the performance of our macros. The macros one can take if something goes wrong are listed below.

- **Originate.** [Ctrl+**o**]. Perform the macro by Ctrl+**o**, and then use Ctrl+**b**. This macro restores the original status of the game saved by the Bac**K**up, i.e., saved by ctrl-**k**.
- **RandM.** [Ctrl+**m**]. The macro Ctrl+m rearranges columns of **Staff Employees**' priority **M** table by random (permutations). N.B. the effect upon staff employees' rankings **M**.
- **RandW.** [Ctrl+**w**]. The macro by Ctrl+w rearranges rows of **clients** priority table **W** by random permutations. N.B. the effect upon client's priority table **W**.

- **Proceed.** [Ctrl+**e**]. While proc**E**eding with macros Rand**M** and Rand**W,** the macro is using random permutations for agency staff employees and client until it generates the priority tables **M** and **W** with minimum mutual risk equal to 4.
- **Blank.** [Ctrl+**u**]. This macro is removing from the list of participants those participants that do not wish to play the game. We call them blank agents. Activate the row-1, or column-A by pointing at employee $\mathbf{m}_{\#\#}$, or client $\mathbf{w}_{\#\#}$ and then perform Ctrl+**u** excluding the chosen participants from playing the game.
- **MAttendee.** [Ctrl+**a**]. Try to m**A**tch [ctrl+**a**] partners by pointing at the cell in the upper block: pink color to the left (or yellow to the right) in the row $\mathbf{w}_\mathbf{i}$ (corresponding to an client) and the column $\mathbf{m}_\mathbf{j}$ (corresponding to a moderator).
- **TrackR.** [Ctrl+**r**]. Visualizes Tracking forwa**R**d. Memorizes the status of *clients*-**W** and *Staff Employees*-**M** rankings to be restored by **TrackB** macro. The effect is invisible, however, it can be used whenever it is appropriate to save the active status of all tables and arrays necessary to restore the status by **TrackB** macro. When the search for quasi-core matchings is performed manually, the effect becomes visible.
- **TrackB.** [Ctrl+**b**] Visualizes Tracking **B**ack. Restores the status of *client*-**W** and *Staff Employees*-**M** rankings memorized by **TrackR** macro.
- **Happiness.** [Ctrl+**p**]. The macro calculates an index of ha**P**piness of the initial tables status.
- **Matching.** [Ctrl+**n**]. The macro rebuilds the matching matching following the matching matching list previously transferred into area "AV24:AO25".
- **Chernoff.** [Ctrl+**q**]. Useful when indicating **by red font** in Excel the status of the Choice Operator **C(X)={argmin}**. Using this macro will help to confirm the validity of the Succession Operator. To clear the status, use Ctrl+**l**.

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VISUALIZATION OF THE MARKETING GAME WITH 20 CLIENTS AND 20 STAFF EMPLOYEES

Legend

Client → Clients' Awards,
Staff → Employees' Awards
Cash → Cashier's Balance

Client nr. 19
Employee nr. 5

Any agreement signed outside the quasi-core will reduce payoffs (including compensation) to all participants, however this will improve the cashier's balance. In this case, the members of the quasi-core will recommend stopping the game.

Client	Staff	Cash	Period												
515€	515€	970€	*1*	19											
				5											
530€	530€	940€	*2*	19	1										
				5	9										
660€	660€	680€	*3*	19	1	18									
				5	9	10									
800€	800€	400€	*4*	19	1	18	6								
				5	9	10	17								
950€	950€	100€	*5*	19	1	18	6	20							
				5	9	10	17	15							
980€	980€	40€	*6*	19	1	18	6	20	17						
				5	9	10	17	15	2						
1.010€	1.010€	20€	*7*	19	1	18	6	20	17	4					
				5	9	10	17	15	2	14					
1.040€	1.040€	80€	*8*	19	1	18	6	20	17	4	11				
				5	9	10	17	15	2	14	12				
1.070€	1.070€	140€	*9*	19	1	18	6	20	17	4	11	15			
				5	9	10	17	15	2	14	12	6			
1.100€	1.100€	200€	*10*	19	1	18	6	20	17	4	11	15	5		
				5	9	10	17	15	2	14	12	6	4		
1.130€	1.130€	260€	*11*	19	1	18	6	20	17	4	11	15	5	10	
				5	9	10	17	15	2	14	12	6	4	7	
1.160€	1160€	320€	*12*	19	1	18	6	20	17	4	11	15	5	10	8
				5	9	10	17	15	2	14	12	6	4	7	3

Period 12 deals $\alpha_1, \alpha_2 \ldots \alpha_{12}$ of participants, which represent stable matching situation, like quasi-core agents in a marketing game on the 6th level of incompatibility of the risk indicator scale. In period 1, the risk score was at level 3.

The total amount $F(X)\cdot\left[|D\boldsymbol{x}| + ½\left(|\mathscr{P}| - |D\boldsymbol{x}|\right)\right] + |\mathscr{P}|$ of rewards + compensations, inclusive goodies, is equal to
{6·[2·12+½·(2·20–2·12)]+2·20}·10€={6·[24+½·16]+40}·10€ =**2.320€**.

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