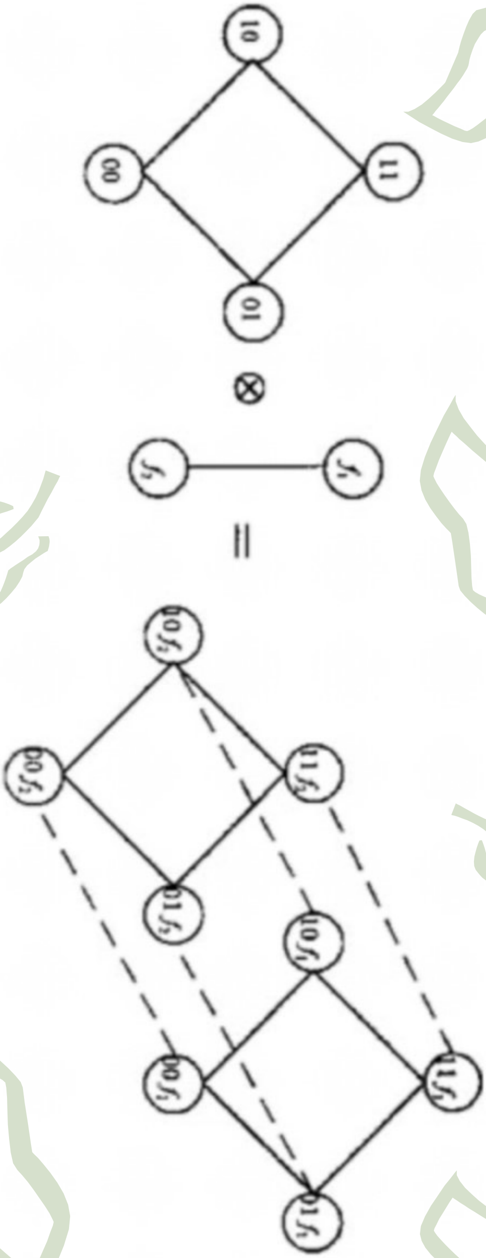

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Extremal Subsystems of Monotonic Systems [i]

Abstract. In the exploration of complex systems, a pivotal aspect involves analyzing specific numerical data to comprehend the system's functioning. This effort often extends to identifying specialized elements or subsystems within the system, discerned by their consistent response to defined 'actions' and intricate 'relations' among homogeneous subsystems. Understanding these nuanced characteristics through rigorous mathematical analysis, elucidating the underlying structure of the system, is crucial, particularly as a foundation for conducting complex or resource-intensive statistical studies. The research explores this basic methodology to identify single-peak sequences that define components of what we call "monotonic systems," where peaks represent "kernels" and "hikes" are depicted as ”stable sets.” Furthermore, we extensively delve into an additional constructive methodology involving two defining sequences within monotonic systems. Through meticulous exploration, we uncover the complex relationship between these defining sequences, ultimately leading to the formulation of the duality theorem. This theorem not only serves as a cornerstone in our understanding but also provides a systematic approach for limiting the search area for kernels and stable sets. In light of this, we present an algorithm designed specifically for the identification of extremal subsystems, namely kernels and stable subsets, within a monotonic system, encapsulated by a certain dual scheme.
Keywords: monotonic; system; matrix; graph; cluster

1. INTRODUCTION

For the study of a complex system, it is often necessary to encounter the problem of analyzing numerical case data about the system functioning. Sometimes based on similar data it is required to show whether in the system there exist special elements or subsystems, reacting in one way to some “actions” as well as “relations” between one-type subsystems. Information on the existence of the indicated peculiarities or on the “structure” of the system under study is necessary, for example, before carrying out extensive or expensive statistical investigation.

Concerning wide application of computational techniques, at the present time, to initial detection of the structure of a system an approach based on various kind of heuristic models is planned (Braverman et al, 1974; McCormik, 1972; Deutch, 1971; Zahn, 1971). For constructing models, many authors start with intuitive formulations of the problem and also with the form of presentation of the initial data (Võhandu, 1964; Терентьев, 1959).

A natural form of presentation the data for the purpose of studying complex systems is that of a graph (Muchnik, 1974). A matrix, for example, a data matrix (Hartigan, 1972) also serves as a widely spread carrier of information. Matrices and graphs easily admit isolation of two minimal structural units of the system: ”elements” and ”connections” between elements.[1] In this paper the notions ”connections” and ”elements” are interrelated in a sufficiently broad fashion. Thus, sometimes it is desirable to consider connections in the form of elements of a system; in this case, it is possible to find more ”subtle” relations in the original system.

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[1] Analogous systems are called systems of interrelated elements in the literature.

Representation of the system in the form of a unique object, comprising elements and connections between them, enables a more precise understanding of the system's structure. This structure entails the organization of system elements into subsystems, delineated by a network of relationships between them. Such a structure may manifest as a natural amalgamation of subsystems into a cohesive whole, delineated by the strength or weakness of interconnections among its elements. This approach finds resonance in the work of Braverman et al. (1971), where the assembly of systems from interconnected elements is expounded upon, revealing assembly as a convenient macro language for expressing system structure.

In system theory, conventional analysis often focuses on direct connections between elements. However, certain scenarios necessitate the consideration of indirect connections as well. These indirect connections are deemed dynamic relations, wherein the degree of interdependence is dictated by the subsystem in which each connection is assessed. Below, we delve into a particular subclass of such dynamic systems what we called as “monotonic systems.”

The foundational property of monotonicity within these systems facilitates the delineation of a system's kernel. This kernel, as initially or primarily indicated, serves as a reflection of the overarching structure of the entire system. Operating within the intrinsic framework of the system, a kernel constitutes a subsystem highly responsive to either positive or negative actions, thus delineating the existence of both positive and negative kernels.

The existence of kernels, which are specialized subsystems, is not left to chance within the mathematical model expounded in this paper; rather, it is a guarantee embedded within the very fabric of the model. The quest to "isolate" these kernels represents a quintessential challenge in the articulation of a "large" system in the parlance of a "small" system – the kernel. In a figurative sense, a kernel of a system embodies a subsystem whose removal invokes profound and irrevocable alterations in the system's properties; it's akin to the system relinquishing its established structure, akin to shedding its skin.

In elucidating the subject matter, the discourse relies on the terminology and symbolism of set theory, a domain accessible to all without necessitating specialized knowledge. However, it warrants attention to the introduction of specific notation, as the framework developed within this paper introduces novel concepts and methodologies. This new apparatus serves as the cornerstone for the exploration and analysis of complex systems, offering insights into their underlying structure and behavior.

2. Examples of Monotonic Systems

In the present paper a monotonic system is defined, to be a system over whose elements one can perform ″positive and ″negative″ actions. In addition, positive actions increase certain quantitative indicators of the functioning of a system while the negative actions decrease those indicators. In the examples considered above the positive action is the addition of an element to a subsystem while the negative action is removing an element from the subsystem; in the third example the converse holds.

In examples, the kernel should possess an intuitive significance. For instance, in citation graphs, a negative kernel would represent publications extensively citing each other, typically authored by individuals from the same scientific school. Conversely, a positive kernel would comprise publications with fewer reciprocal citations, indicating representation from diverse scientific schools.

In transport road networks, the intuitive essence of a kernel should be evident in the following manner. If we consider the elements of a communication network as the transportation routes, then a negative kernel would encompass a set of routes that, on average, experience a significant number of traffic congestions–a sort of consensus among these routes. Conversely, a positive kernel would represent a collection of routes that, on average, encounter fewer traffic congestions, indicating smoother traffic flow.

Alternatively, when the system elements are viewed as the transportation points within the network, a negative kernel would denote a landscape characterized by mutually unreliable points. These points would exhibit a lack of dependability in facilitating transportation connections with one another. On the other hand, a positive kernel would depict a landscape comprising more dependable points, where transportation connections are more reliable and consistent.

I. Examining the complex organization behind the apparently random friend lists found on platforms like Facebook, Linkedin and other social networking media reveals a carefully structured system. Upon closer inspection, it becomes clear that these lists are not random, but rather follow a clearly defined pattern. Each user's friend's list serves as a vital indicator, not only checking connections, but also offering information about mutual acquaintances and potential interests. This gives users the opportunity to make direct connections with new people, seamlessly integrating them into existing social circles.

This process is not simply about expanding one's social circle, but represents a purposeful desire to expand one's social sphere. It is noteworthy that any exclusion of a user from the friends list causes a decrease in the overall score, which means a negative action in the network lexicon. Conversely, adding new connections leads to an increase in the indicator, which means positive interaction with the platform.

These contrasting actions, both negative and positive, are the essence of the formal scheme discussed in this article. By diving deeper into the dynamics of friend lists and related metrics, we gain invaluable insight into the fundamental principles governing social interactions in digital spheres.

In practice, research into social network structures may be conducted incognito, since the identities of the participants and their specific interactions are often irrelevant. Instead of tagging users by name, a simple numbering system is enough to allow chains of actions–both positive and negative–to be built within the network. This approach contributes to a deeper understanding of the complex dynamics of internal relations, allowing researchers to explore different mutual reflections and combinations of interactions when analyzing network structure.

II. This excerpt elaborates on enhancing the efficiency of cellular networks through spatial signal processing and adaptive antennas. It underscores the intricate interplay among antenna arrays, processing algorithms, and resource allocation for maximizing data throughput. By focusing on specific parametric classes of antenna systems, optimization becomes more feasible, allowing for the estimation of benefits from adaptive antennas. The example replicates Shorin et al.'s 2016 study for antennas distribution. The study also introduces a novel algorithm facilitating Monotonic Systems efficient grouping of antennas based on angular diversity, ensuring optimal resource utilization.

The introduction of spatial signal processing technology and adaptive antennas makes it possible to significantly (manifold) increase the throughput of the radio channel due to the active use of the resource associated with the capabilities of spatial signal selection.

In the context of cellular networks, optimizing adaptive spatial processing entails a shift from traditional approaches to achieving maximum throughput for a radio channel connecting numerous spatially dispersed subscribers with a serving base station. This shift emphasizes the interdependence of the antenna array, spatial processing algorithm, radio channel resource distribution algorithm, and data exchange algorithms, forming a unified hardware and software module dedicated to solving the transmission problem. While the optimal design of antenna arrays and algorithms remains a question, practical simplifications can be made by constraining antenna systems to specific parametric classes, such as ring homogeneous structures with adjustable placement radii and radiation pattern widths.

The following approach facilitates optimization and allows estimation of the benefits derived from using adaptive antennas, often through simulation. Furthermore, the proposed algorithm in this article introduces a "mode with reverse extraction of elements from groups," enabling the creation of minimal clusters with desired angular diversity levels. Additionally, this mode facilitates the distribution of subscribers in favorable locations across multiple groups, maximizing the utilization of available radio channel resources.

In the particular scenario of the "Monotone System" being addressed, the algorithm outlined in this article offers a precise solution. This algorithm introduces a "mode with reverse extraction of elements from groups," which serves a dual purpose. Firstly, it enables the creation of the fewest possible groups or clusters while maintaining a specified level of angular diversity. Secondly, it facilitates the simultaneous allocation of individual subscribers situated in more favorable locations across multiple groups. This approach ensures optimal utilization of the available resources within the radio channel, maximizing efficiency and performance.

III. Let's consider a scenario where there exists a network of transportation exchanges or nodes, denoted as landscape **W**, interconnected by two-sided roads. In the absence of direct transportation between these nodes within this road system, transit transportation can be organized. Over a long period of observation, if such a pattern of operation persists regardless of the presence of direct transport links, it is possible to assess the efficiency of transportation by measuring the average frequency of traffic jams when establishing transportation between these nodes within a standard unit of time. Essentially, to characterize the reliability of establishing transportation between each node in a system W, one can use the average number of traffic congestions experienced by connecting to at least one destination node in the system over a given period of time. It is obvious that these quantitative indicators, namely the feasibility of transportation over a given period of time and the characteristics of the guarantees provided, are applicable only within each subsystem of the road network **W**.

The proposed model exhibits several inherent characteristics. Any interruption in the flow of transportation along a two-sided route amplifies the average number of traffic congestions among all other transportation points, while the introduction of a new route conversely diminishes this average. This dynamics correlates with an increase or decrease in the load on facilitating transit transportation within the transport communications network.

Similarly, when activity is scaled back at any transportation point within a given subsystem, the unreliability of all points within that subsystem escalates. Conversely, the addition of a new transportation point to the subsystem reduces this unreliability. These observations mirror the behavior of monotonic systems discussed earlier, affirming that the model governing transportation roads adheres to the principles of a monotonic system.

IV. In the exploration of academic research, various scientific disciplines utilize graphs of cited publications, as outlined by Налимов and Мульченко in 1969. These graphs are directed and a-cyclic, reflecting the nature of scholarly citations where authors can only cite papers that have already been published. It is reasonable to conceptualize the set of publications, denoted as **W**, as a system where information exchange occurs through citations.

Within this framework, considering a subset of publications from the entire set **W** allows us to characterize each publication based on the number of bibliographical references within that subset. When a publication is removed from the subset, this quantitative measure of information exchange within the subset diminishes. Conversely, adding a publication to the subset enhances this evaluation for all publications within the subset. Hence, the citation system represented by these graphs exhibits monotonic behavior. In a related context, Trybulets (1970) highlights an intriguing example where a directed graph inadvertently illustrates the concept of a monotonic system

V. In the n-dimensional vector space let there be given **N** vectors. For each pair of vectors x and y one can define in many ways a distance $\rho(x, y)$ between these vectors (i.e., to scale the space). Let us assume that the set of given vectors forms an unknown system **W**. For every vector in an arbitrary subsystem of **W** we calculate the sum of distances to all vectors situated inside the selected subsystem. Thus, with the respect to each subsystem of **W** and each vector situated inside that subsystem, a characteristic sum of distances is defined, which can be different for different subsystems. It is not difficult to establish the following property of the set of sums of distances. Because of removing a vector from the subsystem the sums computed for the remaining vectors decrease while because of adding a vector to the subsystem they increase. A similar property of sums for every subsystem of system **W** is called in this paper the monotonicity property and a system **W** having such a property is called a monotonic system.

3. Description of a Monotonic System

One considers some system W consisting of a finite number of elements,[3] i.e., $|W| = N$, where each element α of the system W plays a well-defined role. It is supposed that the states of elements α of W are described by definite numerical quantities characterizing the "significance" level of elements α for the operation of the system as a whole and that from each element of the system one can construct some discrete actions.

We reflect the intrinsic dependence of system elements on the significance levels of individual elements. The intrinsic dependence of elements can be regarded in a natural way as the change, introducible in the significance levels of elements β, rendered by a discrete action produced upon element α.

We assume that the significance level of the same element varies as a result of this action. If the elements in a system are not related with each other in any way, then it is natural to suppose that the change introduced by element α on significance β (or the influence of α on β) equals zero.

We isolate a class of systems, for which global variations in the significance levels introduced by discrete actions on the system elements bears a monotonic character.

Definition. By a monotonic system, we understand a system, for which an action realized on an arbitrary element α involves either only decrease or only increase in the significance levels of all other elements.

In accordance with this definition of a monotonic system two types of actions are distinguished: type $\oplus$ and type $\ominus$. An action of type $\oplus$ involves increase in the significance levels while $\ominus$ involves decrease.

The formal concept of a discrete action on an element α of the system W and the change in significance levels of elements arising in connection with it allows us to define on the set of remaining elements of W an uncountable set of functions whenever we have at least one real significance function $\pi : W \to D$ (D being the set of real numbers).

Indeed, if an action is rendered on element α, the starting from the proposed scheme one can say that function π is mapped into π_α^+ or π_α^- according as a the action $\oplus$ or $\ominus$. Significance of system elements is redistributed as action on element α changes from function π to π_α^+ (π_α^-) or, otherwise, the initial collection of significance levels $\{\pi(\partial) \mid \partial \in W\}$ changes into a new

[3] If W is a finite set, then $|W|$ denotes the number of its elements.

collection $\{\pi_\alpha^+(\partial) \mid \partial \in W\}$.[4] Clearly, if we are given some sequence $\alpha_1, \alpha_2, \alpha_3, ...$ of elements of W (arbitrary repetitions and combinations of elements being permitted) and the binary sequence $+,-,+,...$, then by the usual means one can define the functional product of functions $\pi_{\alpha_1}^+, \pi_{\alpha_2}^-, \pi_{\alpha_3}^+$ in the form $\pi_{\alpha_1}^+ \pi_{\alpha_2}^- \pi_{\alpha_3}^+$.

The construction presented allows us to write the property of monotonic systems in the form of the following basic inequalities:

$$\pi_\alpha^+(\partial) \geq \pi(\partial) \geq \pi_\alpha^-(\partial) \tag{1}$$

for every pair of elements $\alpha, \partial \in W$, including the pairs α, α or ∂, ∂.

Let there be given a partition of set W into two subsets, i.e., $H \cup \overline{H} = W$ and $H \cap \overline{H} = \varnothing$. If we subject the elements $\alpha_1, \alpha_2, \alpha_3, ... \in \overline{H}$ to positive actions only, then by the same token on set W there is defined some function $\pi_{\alpha_1}^+ \pi_{\alpha_2}^- \pi_{\alpha_3}^+ ...$, which can be regarded as defined only on the subset H of W.[5]

If from all possible sequences of elements of set $\overline{H}$ we select a sequence $\left\langle \alpha_1, \alpha_2, ..., \alpha_{|\overline{H}|} \right\rangle$,[6] where α_i are not repeated, then on the set H the function $\pi_{\alpha_1}^+ \pi_{\alpha_2}^+ ...$ is induced ambivalently.

We denote this function $\pi^+ H$ and call it a standard function. We shall also refer to the function thus introduced as a credential function and to its value on an element as an α credential. In accordance with this terminology the set $\{\pi^+ H(\alpha) \mid \alpha \in H\}$, which is denoted by $\Pi^+ H$ is called a credential collection given on the set H or a credential collection relative to set H. Let us assume that we are given a set of credential collections $\{\Pi^+ H \mid H \subseteq W\}$ on the set of all possible subsystems $P(W)$ of system W. The number of all possible subsystems is $|P(W)| = 2^{|W|}$.

Instead of considering a standard function for positive actions $\pi_{\alpha_1}^+ \pi_{\alpha_2}^+ ...$ one can consider a similar function for negative actions $\pi^- H$. Thus, one defines single credential collection $\Pi^- H = \{\pi^- H(\alpha) \mid \alpha \in H\}$ and the aggregate of credential collections $\{\Pi^- H \mid H \subseteq W\}$ by an exact analogy.

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[4] Functions π, π_α^+ and π_α^- are defined on the whole set W and, consequently, $\pi_\alpha^+(\partial)$ and $\pi_\alpha^-(\partial)$ are defined.

[5] We are not interested in significance levels obtained as a result of operations on elements of $\overline{H}$ onto the same set $\overline{H}$.

[6] Here symbols $\langle , \rangle$ are used to stress the ordered character of a sequence of $\overline{H}$.

Let us briefly summarize the above construction. Starting with some real function π defined on a finite set W and using the notion of positive and negative actions on elements of system W, one can construct two types of aggregate collections $\Pi^{+}H$ and $\Pi^{-}H$ defined on each of the H of subsets of W. Each function from the aggregate (credential collection) is constructed by means of the complement to H, equaling $W \setminus H$, and a sequence $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{H}|} \right\rangle$ of distinct elements of the set $\overline{H}$. For this actions of types $\oplus$ and $\ominus$ are applied to all elements of set $\overline{H}$ in correspondence with the ordered sequence $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{H}|} \right\rangle$ in order to obtain $\Pi^{+}H$ and $\Pi^{-}H$ respectively.

Credential collections/arrays concept of $\Pi^{+}H$ and $\Pi^{-}H$ needs refinement. The definition given above does not taken into account the character of dependence of function πH on the sequence of actions realized on the elements of set $\overline{H}$.[7] Generally speaking, credential collection $\Pi^{+}H(\Pi^{-}H)$ is not defined uniquely, since it can happen that for different orderings of set $\overline{H}$ we obtain different function πH.

In order that credential collection $\Pi^{+}H$ $(\Pi^{-}H)$ be uniquely defined by subset H of the set W it is necessary to introduce the notion of commutability of actions.

Definition. An action of type $\oplus$ or $\ominus$ is called commutative for system W if for every pair of elements $\alpha, \beta \in W$ we have

$$\pi_{\alpha}^{+}\pi_{\beta}^{+} = \pi_{\beta}^{+}\pi_{\alpha}^{+},\ \pi_{\alpha}^{-}\pi_{\beta}^{-} = \pi_{\beta}^{-}\pi_{\alpha}^{-}$$

In this case it is easy to show that the values of function πH on the set H do not depend on any order defined for the elements of the set $\overline{H}$ by sequence $\langle \alpha_1, \alpha_2, \ldots \rangle$. The proof can be conducted by induction and is omitted.

Thus, for commutative actions the function $\pi^{+}H$ $(\pi^{-}H)$ is uniquely determined by a subset of W.

In concluding this section, we make one important remark of an intuitive character. As is obvious from the above-mentioned definition of aggregates of credentials collection of type $\oplus$ and $\ominus$, the initial credential collection serves as the basic constructive element in their construction. The initial credential collection is a significance function defined on the set of system elements before

[7] In the sequel, if sign "–" or "+" is omitted from our notation, then it is understood to be either "–" or "+"

the actions are derived from the elements. In other words, it is the initial state of the system fixed by credential collection ΠW. It is natural to consider only those aggregates of credential collections that are constructed from an initial ⊕ collection, which is the same as the initial ⊖ collection. The dependence indicated between ⊕ and ⊖ credential collections is used considerably for the proof of the duality theorem in the second part of this paper.

4. Extremal Theorems. Structure of Extremal Sets

Let us consider the question of selecting a subset from system W whose elements have significance levels that are stipulated only by the internal "organization" of the subsystem and are numerically large or, conversely, numerically small. Since this problem consists of selecting from the whole set of subsystems $P(W)$ a subsystem having desired properties, therefore it is necessary to define more precisely how to prefer one subsystem over another, see also Muchnik and Shvartser (1990).

Let there be given aggregates of credential collections $\{\Pi^{+}H \mid H \subseteq W\}$ and $\{\Pi^{-}H \mid H \subseteq W\}$. On each subset there are defined the following two functions:

$$F_{+}(H) = \max_{\pi \in H} \pi^{+}H(\alpha), \ F_{-}(H) = \min_{\pi \in H} \pi^{-}H(\alpha).$$

Definition of Kernels. By kernels of set W we call the points of global minimum of function F_{+} and of global maximum of function F_{-}.

A subsystem, on which F_{+} reaches a global minimum is called a ⊕ kernel of the system W, while a subsystem on which F_{-} reaches a global maximum, is called ⊖ kernel. Thus, in every monotonic system the problem of determining ⊕ and ⊖ kernels is raised.

With the purpose of intuitive interpretation as well as with the purpose of explaining the usefulness of the notion of kernels introduced above we turn once again to the examples of citation graphs and telephone commutation networks.

The definition of the kernel can be formulated using the levels of significance of the elements of the system, that is: the ⊕ kernel is a subsystem of a monotonic system, for which the maximum level among the levels of significance is determined only by the internal organization of the system is the minimum, and the ⊖ kernel is the subsystem for which the minimum level among the same significance levels is the maximum.

The definition of a kernel accords with the intuitive interpretation of a kernel in citation graphs and telephone commutation networks. Thus, in citation graphs a ⊕ kernel is a subset (subsystem) of publications, in which the longest list of bibliographical titles is at the same time very short; though not inside the subset, but among all possible subsets of the selected set of publications (among the very long lists). If in our subset of publications a very short list of bibliographical titles is at the same time very long among the very short ones relative to all the subsets, then it is a ⊖ kernel of the citation graph. It is clear that a ⊖ kernel publications cite one another often enough, since for each publication the list of bibliographical titles is at any rate not less than a very short one while a very short list is nevertheless long enough. In a ⊕ kernel the same reason explains why in this subset one must find representatives of various scientific schools.

In telephone commutation networks, one can consider two types of system elements – lines of connections and points of connections. In a system consisting of lines, a ⊖ kernel turns out to be a subset of lines, for which the lines with the least number of traffic congestions in that subset are at the same time the lines with the greatest number of traffic congestions among all possible sets of lines. This means that at least the number of traffic congestions stipulates only by the internal organization of a sub-network of lines of a ⊖ kernel is not less than the number of traffic congestions for lines with the smallest number of traffic congestions and, besides, this number is large enough. Hence one can expect that the number of traffic congestions for lines of a ⊖ kernel is sufficiently large. Similarly one should expect a small number of traffic congestions for lines of a ⊕ kernel. Formulation for ⊕ and ⊖ kernels for points of connection is exactly the same as for the lines and is omitted here.

Before stating the theorems, we need to introduce some new definitions and notations. Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1} \rangle$ be an ordered sequence of distinct elements of set W, which exhausts the whole of this set, i.e., $k = |W|$. From sequence $\overline{\alpha}$ we construct an ordered sequence of subsets of W in the form $\Delta_{\overline{\alpha}} = \langle H_0, H_1, \ldots, H_{k-1} \rangle$ with the help of the following recurrent rule $H_0 = W$, $H_{i+1} = H_i \setminus \{\alpha_i\}$; $i = 0,1,\ldots,k-2$ [9]

Definition. Sequence $\overline{\alpha}$ of elements of W is called a defining sequence relative to the aggregate of credentials collections $\{\Pi^{-}H \mid H \subseteq W\}$ if there exists in sequence $\Delta_{\overline{\alpha}}$, a subsequence of sets $\Gamma_{\overline{\alpha}} = \langle \Gamma_0^{-}, \Gamma_1^{-}, \ldots, \Gamma_p^{-} \rangle$, such that:

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[9] Sign $\setminus$ denotes the subtraction operation for sets.

a) credential $\pi^{-}H_i(\alpha_i)$ of an arbitrary element α_i in sequence $\overline{\alpha}$, belonging to set Γ_j^{-} but not belonging to set Γ_{j+1}^{-} is strictly less than values of $F_{-}(\Gamma_{j+1})$; [10]

b) in set Γ_p^{-} there does not exist a proper subset L, which satisfies the strict inequality $F_{-}(\Gamma_p) < F_{-}(L)$.

A sequence $\overline{\alpha}$ with properties a) and b) is denoted by $\overline{\alpha}_{-}$. One similarly defines a sequence $\overline{\alpha}_{+}$.

c) arbitrary element α_i in sequence $\overline{\alpha}$, belonging to set Γ_j^{+} but not belonging to set Γ_{j+1}^{+} is strictly greater than values of $F_{+}(\Gamma_{j+1})$;

d) in set Γ_q^{+} there does not exist a proper subset L, which satisfies the strict inequality $F_{+}(\Gamma_q) > F_{+}(L)$.

Definition. Subset H_{+}^{*} of set W is called definable if there exists a defining sequence $\overline{\alpha}_{+}$ such that $H_{+}^{*} = \Gamma_q^{+}$.

Definition. Subset H_{-}^{*} of set W is called definable if there exists a defining sequence $\overline{\alpha}_{-}$ such that $H_{-}^{*} = \Gamma_p^{-}$.

Below we formulate, but do not prove, a theorem concerning properties of points of global maximum of function F_{-}. The proof is adduced in Appendix 1. A similar theorem holds for function F_{+}. In Appendix 1 the parallel proof for function F_{+} is not reproduced. The corresponding passage from the proof for F_{-} to that of F_{+} can be effected by simple interchange of verbal relations "greater than" and "less than", inequality signs "≥" and "≤", ">", "<" as well as by interchange of signs "+" and "–". The passage from definable set H_{+}^{*} to H_{-}^{*} and from definition of sequence $\overline{\alpha}_{+}$ and $\overline{\alpha}_{-}$, is affected by what has just been said.

Theorem 1. On a definable set H_{-}^{*} function F_{-} reaches a global maximum. There is a unique definable set H_{-}^{*}. All sets, on which a global maximum is reached, lie inside the definable set H_{-}^{*}.

[10] Here and everywhere, for simplification of expression, where it is required, the sign "–" or "+" is not used twice in notations. We should have written $F_{-}(\Gamma_{j+1}^{-})$ or $F_{+}(\Gamma_{j+1}^{+})$.

Theorem 2. On a definable set H^*_+ function F_+ reaches a global minimum. There is a unique definable set H^*_+. All sets, on which a global minimum is reached, lie inside the definable set H^*_+.

In the proof of Theorem 1 (Appendix 1) it is supposed that definable set H^*_- exists. It is natural that this assumption, in turn, needs proof. The existence of H^*_- is secured by a special constructive procedure. [11]

The proof of Theorem 2 is completely analogous to the proof of Theorem 1 and is not adduced in Appendix 1. We present a theorem, which reflects a more refined structure of kernels of W as elements of the set $P(W)$ of all possible subsets (subsystems) of set W.

Theorem 3. The system of all sets in $P(W)$, on which function F_- (F_+) reaches maximum (minimum), is closed with the respect to the binary operation of taking union of sets.

The proof of this theorem is given in Appendix 2 and only for the function F_-. The assertion of the theorem for F_+ is established similarly.

Thus, it is established that the set of all ⊕ kernels (⊖ kernels) forms a closed system of sets with respect to the binary operation of taking the unions. The union of all kernels is itself a large kernel and, by the statements of Theorems 1 and 2, is a definable set.

5. Routine of Finding the Kernels

In preceding sections, we established the fundamental approach for selecting singular subsystems within monotonic systems, specifically identifying kernels with extremal properties. At the core of this method lies the notion of a 'definable set,' as delineated by Mullat in 1971. In our framework, a definable set represents the largest kernel within a monotonic system of interconnected elements. Back in 1971, we introduced the concept of a definable set through the utilization of defining $\overline{\alpha}_-$ and $\overline{\alpha}_+$ sequences within the system.

Subsequently, we tackled the issue of identifying defining sequences, offering constructive solutions in the form of algorithms. The key attributes of these defining sequences, generated according to predefined routines, and encompassing the entirety of system elements **W**, are delineated by a theorem.

[11] This procedure will be presented in the second part of the article, since here only the extremal properties of kernels and the structure of the set of kernels are established.

We will delve into the intricate relationship between two defining sequences, denoted as $\overline{\alpha}_-$ and $\overline{\alpha}_+$. While one might intuitively consider obtaining $\overline{\alpha}_+$ by simply reversing the order of $\overline{\alpha}_-$, this assumption doesn't universally hold true. However, we can make a more nuanced assertion based on the discrete operations ⊕ and ⊖ on the elements of system W, as defined by Mullat in 1976. This assertion manifests as a duality theorem, which we shall expound upon shortly.

Under the auspices of this duality theorem, the algorithms elucidated for constructing defining sequences serve to significantly narrow the scope of search for both ⊕ and ⊖ kernels within system W. The algorithm delineating this restriction of the search domain is presented in the form of a constructive routine.

Now, let's dissect the routine for constructing an ordered sequence α comprising all elements of W, succinctly known as the Kernel Searching Routine (KSR). This routine plays a pivotal role in our methodology, facilitating the systematic identification and organization of system elements for further analysis and manipulation.

This routine consists of rules of generation and scanning of an ordered series of ordered sets $\langle\overline{\beta}_j\rangle$ (sequences); here j varies from zero to a value p, which is automatically determined by the rules of the routine, whereas the elements of each sequence $\overline{\beta}_j$ are selected from the set W [12].

This series $\langle\overline{\beta}_j\rangle$ constructed by this rule forms a numerical sequence of thresholds $\langle u_j\rangle$ and a sequence of sets $\langle\Gamma_j\rangle$. On the other hand the sequence of thresholds governs the transactions from $\overline{\beta}_{j-1}$ to $\overline{\beta}_j$ in the chain $\langle\overline{\beta}_j\rangle$, and the sequence $\langle\Gamma_j\rangle$ terminates with a set, which is definable.

In the description of a rule we use the operation of extending a sequence $\overline{\beta}_j$ by adjoining to it another sequence $\overline{\gamma}$. This operation is symbolically expressed by $\overline{\beta} \leftarrow \langle\overline{\beta},\overline{\gamma}\rangle$. This rule of construction of the sequence $\overline{\alpha}$ of all elements of the set W can be described stages: by step **Z** and **R**.

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[12] Let us recall that in a) the brackets $\langle,\rangle$ denoted an ordered set; in the case under consideration they denote an ordered set of ordered sets $\overline{\beta}_j$.

Z. In the set W we have found an element μ_0 for which $\pi^- W(\mu_0) = \min_{\delta \in W} \pi^- W(\delta) = F_-(W)$; we are constructing a defining sequence $\overline{\alpha}_-$. The construction of $\overline{\alpha}_+$ is entirely similar and therefore not presented here. We shall only indicate where it is necessary to invert the sign of inequalities, and where the search for an element with the minimal credential must be replaced by search for an element with maximal credential, so as to be able to construct $\overline{\alpha}_+$. Thus the construction of $\overline{\alpha}_+$, the element μ_0 is obtained from $\pi^+ W(\mu_0) = \max_{\delta \in W} \pi^+ W(\delta) = F_+(W)$ condition. We shall write $u_0 = \pi^- W(\mu_0)$, $\overline{\alpha} = \langle \mu_0 \rangle$ and the set $\Gamma_0 = W$. We select a subset of elements γ from W such that $\pi^- W \setminus \overline{\alpha}(\gamma) \leq u_0$. The construction of $\overline{\alpha}_+$ requires the selection of such γ that $\pi^+ W \setminus \overline{\alpha}(\gamma) \geq u_0, u_0 = \pi^+ W(\mu_0)$. After that we order the elements in a certain manner (which can be arbitrary selected). The thus-obtained ordered set is denoted by $\overline{\gamma}$. Let us write $\overline{\beta}_0 = \overline{\gamma}$.

R. We construct a recursive routine for extending the sequences $\overline{\alpha}$ and $\overline{\beta}_0$. Here we denote by $\beta_0(i)$ the i-th element of the sequence $\overline{\beta}_0$. We specify one after another the elements of the sequence $\overline{\beta}_0$. At each instant of specification we extend the sequence $\overline{\alpha}$ by the elements from $\overline{\beta}_0$ of the sequence fixed at this instant. In accordance with the symbolic notation of the operation of extension of a sequence $\overline{\alpha}$, we perform at each instant t of specification the operation $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \beta_0(t) \rangle$. Suppose that all the elements of $\overline{\beta}_0$ up to $\beta_0(i-1)$ inclusive have been fixed. Then the sequence $\overline{\alpha}$ will have the form $\langle \mu_0, \beta_0(1), \beta_0(2), ..., \beta_0(i-1) \rangle$, which corresponds to the symbolic notation of the operation of extension of the sequences $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \beta_0(1), \beta_0(2), ..., \beta_0(i-1) \rangle$ in the case that $\overline{\alpha}$ inside the brackets consists of one element μ_0. Let us consider an element $\beta_0(i-1)$ of the sequence $\overline{\beta}_0$. At the instant of specification of the element $\beta_0(i-1)$ we decide during the above-mentioned operation of extension of $\overline{\alpha}$ also about any further extension or about stopping the extension of the sequence $\overline{\beta}_0$. We must check the following three conditions:

a) In the set $W \setminus \overline{\alpha}$ there exist elements such that $\pi^- W \setminus \overline{\alpha}(\gamma) \leq u_0$ In constructing $\overline{\alpha}_+$, this condition is replaced by $\pi^+ W \setminus \overline{\alpha}(\gamma) \geq u_0$;

b) the element $\beta_0(i)$ is defined for the sequence $\overline{\beta}_0$. By assumption an element $\beta_0(i)$ to be defined for a sequence $\overline{\beta}_0$ if the sequence $\overline{\beta}_0$ has an element with an ordinal number i. Otherwise the element $\beta_0(i)$ is not defined. There can be four cases of fulfillment or no fulfillment of these conditions. In two cases, when the first condition is satisfied, irrespective of whether or not the second condition holds, the sequence $\overline{\beta}_0$ will be extended. This means that the set of elements γ in $W \setminus \overline{\alpha}$ specified by the first condition is ordered in the form of sequence $\overline{\gamma}$. The sequence $\overline{\beta}_0$ is extended in accordance with the formula $\overline{\beta}_0 \leftarrow \langle \overline{\beta}_0, \overline{\gamma} \rangle$. In case when the first condition is not satisfied, whereas the second condition is satisfied, we shall fix the element $\beta_0(i)$ and at the same time extend the sequence $\overline{\alpha}$, i.e., $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \beta_0(i) \rangle$, and proceed to new recursion stage. In case neither the first nor the second condition holds, the sequence $\overline{\beta}_0$ will not be extended nor the last fixed element in the sequence $\overline{\beta}_0$ will be the element $\beta_0(i-1)$. Suppose that we have fixed all the elements of the sequence $\overline{\beta}_j$. By that time we have constructed a sequence $\overline{\alpha}$. Let us consider the set $W \setminus \overline{\alpha}$ and the credential system $\Pi^{-}W \setminus \overline{\alpha}$. We shall find an element in $\Pi^{-}W \setminus \overline{\alpha}$ on which the minimum is reached in the credential system $\Pi^{-}W \setminus \overline{\alpha}$. The obtained element is denoted by μ_{j+1}. We obtain $\overline{\alpha}_{+}$ the element μ_{j+1} from the $\pi^{+}W \setminus \overline{\alpha}(\mu_{j+1}) = \max_{\delta \in W \setminus \overline{\alpha}} \pi^{+}W \setminus \overline{\alpha}(\delta) = F_{+}(W \setminus \overline{\alpha})$ condition:. Thus, $\pi^{-}W \setminus \overline{\alpha}(\mu_{j+1}) = F_{-}(W \setminus \overline{\alpha})$. Let us write $u_{j+1} = \pi^{-}W \setminus \overline{\alpha}(\mu_{j+1})$, and for the set $\Gamma_{j+1} = W \setminus \overline{\alpha}$; then we supplement the sequence $\overline{\alpha}$ by the element μ_{j+1}, i.e., $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \mu_{j+1} \rangle$. In the same way as during the zero step, we select a subset of elements γ from $W \setminus \overline{\alpha}$ such that $\pi^{-}W \setminus \overline{\alpha}(\gamma) \leq u_{j+1}$. Here we select for $\overline{\alpha}_{+}$ a set of elements γ such that $\pi^{+}W \setminus \overline{\alpha}(\gamma) \geq u_{j+1}$. The selected set can be ordered in any manner. The ordered set is denoted by $\overline{\gamma}$. The set $\overline{\beta}_{j+1}$ is assumed to be equal to $\overline{\gamma}$.

c) By analogy with previous b) the recursion step will be described as a recursion routine. At this stage we also use the rule of extension of the sequences $\overline{\alpha}$ and $\overline{\beta}_{j+1}$. Suppose that we have fixed all elements of $\overline{\beta}_{j+1}$ up to $\beta_j(i-1)$ inclusive. Then the sequence $\overline{\alpha}$ will have the form $\overline{\alpha} = \langle \overline{\alpha}, \mu_{j+1}, \beta_j(1), ..., \beta_j(i-1) \rangle$, where $\overline{\alpha}$ denotes the sequence $\overline{\alpha}$ obtained at the instant of fixing all the elements of $\overline{\beta}_j$, or, to rephrase, the sequence $\overline{\alpha}$ prior to the $(j+1)$-th step. The last equation corresponds to the symbolic operation of extension of the sequence $\overline{\alpha} = \langle \overline{\alpha}, \mu_{j+1}, \beta_j(1), ..., \beta_j(i-1) \rangle$ in the case that $\overline{\alpha}$ inside the brackets denotes the sequence $\langle \overline{\alpha}, \mu_{j+1} \rangle$. Let us consider an element $\beta_{j+1}(i-1)$ of the sequence $\overline{\beta}_{j+1}$. At the instant of fixing the element $\beta_{j+1}(i-1)$ we decide about a further extension or about stopping the extension of the sequence $\overline{\beta}_{j+1}$. For this purpose we consider the credential system $\Pi^{-} W \setminus \overline{\alpha}$ and we check two conditions:

1) The set $W \setminus \overline{\alpha}$ contains elements γ such that $\pi^{-} W \setminus \overline{\alpha}(\gamma) \leq u_{j+1}$ For constructing $\overline{\alpha}_{+}$ we must take elements γ such that $\pi^{+} W \setminus \overline{\alpha}(\gamma) \geq u_{j+1}$;

2) the element $\beta_{j+1}(i)$ is defined for the sequence $\overline{\beta}_{j+1}$.

By analogy with the step **Z**, we find that the sequence $\overline{\beta}_{j+1}$ is extended in two cases in which the first condition is satisfied irrespective of whether or not the second condition holds. The set of elements γ in $W \setminus \overline{\alpha}$ specified by the first condition is ordered in the form of a sequence $\overline{\gamma}$. The sequence $\overline{\beta}_{j+1}$ is extended in accordance with the formula $\overline{\beta}_{j+1} \leftarrow \langle \overline{\beta}_{j+1}, \overline{\gamma} \rangle$. In the case that the first condition does not hold, whereas the second condition is satisfied, the element $\beta_{j+1}(i)$ will be fixed and at the same time we extend the sequence $\overline{\alpha}$, i.e., $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \beta_{j+1}(i) \rangle$, and after that we proceed again in accordance with the rules of Stage 2 of the recursion routine of extension of the sequence $\overline{\beta}_{j+1}$. In the case that neither the first, nor the second condition holds, the sequence $\overline{\beta}_{j+1}$ will not be extended, and the last fixed element of the sequence $\overline{\beta}_{j+1}$ will be the element $\beta_{j+1}(i-1)$. At some step p the sequence $\overline{\alpha}$ will exhaust the entire set of elements W.

Theorem 4. A sequence $\overline{\alpha}$ constructed on the basis of a collection of credential system $\{\Pi^{-}H \mid H \subseteq W\}$ is a defining sequence $\overline{\alpha}_{-}$, whereas a sequence $\overline{\alpha}$ constructed on the basis of $\{\Pi^{+}H \mid H \subseteq W\}$ is a defining sequence $\overline{\alpha}_{+}$.

The first part of the theorem (for $\overline{\alpha}_{-}$) is proved in Appendix 3. The second part (for $\overline{\alpha}_{+}$) can be proved in the same way.

NB1. Let us note that a sequence $\overline{\alpha}$ constructed by KSR rules has somewhat stronger properties than required in obtaining a defining sequence. More precisely, there does not exist a proper subset L for $j = 0,1,...,p-1$ such that $\Gamma_j \supset L \supset \Gamma_{j+1}$ and $F_{-}(\Gamma_j) < F_{-}(L)$. This is not required for obtaining a defining sequence $\overline{\alpha}_{-}$ ($\overline{\alpha}_{+}$). The corresponding proof is not given here.

NB2. Let us note another circumstance. With the aid of the kernel-searching routine it is possible to effectively find (without scanning) the largest kernel, i.e., a definable set. It is not possible to find an individual kernel strictly included in a definable set (if the latter exists) by constructing a defining sequence.

6. Duality Theorem

Let us establish a relationship between the defining sequences $\overline{\alpha}_{-}$ and $\overline{\alpha}_{+}$ of a system W.

Theorem 5. Let $\overline{\alpha}_{-}$ and $\overline{\alpha}_{+}$ be defining sequences of the set W with respect to the collection of credential system $\{\Pi^{-}H \mid H \subseteq W\}$, $\{\Pi^{+}H \mid H \subseteq W\}$ respectively. Let $\langle \Gamma_j^{-} \rangle$ be the subsequence of the sequence $\Delta_{\overline{\alpha}_{-}}$ $(j = 0,1,...,p)$ needed in the determination of $\overline{\alpha}_{-}$, and let $\langle \Gamma_j^{+} \rangle$ be the corresponding subsequence of the sequence $\Delta_{\overline{\alpha}_{+}}$ $(j = 0,1,...,q)$.

Hence if for an m and a n we have

$$F_{+}(\Gamma_n^{+}) = F_{-}(\Gamma_m^{-}), \tag{2}$$

then $\Gamma_m^{-} \subseteq W \setminus \Gamma_{n+1}^{+}$, $\Gamma_n^{+} \subseteq W \setminus \Gamma_{m+1}^{-}$. If

$$F_{+}(\Gamma_n^{+}) < F_{-}(\Gamma_m^{-}) \text{ [2]}, \tag{3}$$

then $\Gamma_m^{-} \subseteq W \setminus \Gamma_n^{+}$, $\Gamma_n^{+} \subseteq W \setminus \Gamma_m^{-}$.

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[2] In the following, the + and – sign will not be used twice in notation. This rule applies also to Appendices 1 and 2

This theorem is important from two points of view. Firstly, under the conditions (2) and (3) there exists a relationship between an $\overline{\alpha}_-$ sequence and $\overline{\alpha}_+$. This relationship consists in the fact that elements of $\overline{\alpha}_+$ which are at the "beginning" and form either the set $W \setminus \Gamma^+_{n+1}$ or the set $W \setminus \Gamma^+_n$ will include all the elements of the set Γ^-_m that are at the "end" of $\overline{\alpha}_-$. The same applies also to sets $W \setminus \Gamma^-_{m+1}$ or $W \setminus \Gamma^-_m$ which are at the beginning of $\overline{\alpha}_-$, since they include in a similar way the set Γ^+_n. In other words, the theorem states that the sequence $\overline{\alpha}_+$ does not differ "very much" (under certain conditions) from the sequence, which is the inverse to $\overline{\alpha}_-$.

Let us note that the conditions (2) and (3) are sufficient conditions, and it can happen that actual monotonic systems satisfying these conditions do not exist. Nevertheless, in the third part of this article, we shall describe actual examples of such systems.

7. Kernel Search Routine Based on Duality Theorem

We just noted that a defining sequence $\overline{\alpha}_+$ differs "slightly" from the inverse sequence of $\overline{\alpha}_-$. For elucidating the possibility of a search for kernels on the basis of the duality theorem, let us rephrase the latter. This assertion can be formulated as follows: at the beginning of the sequence $\overline{\alpha}_+$ we often encounter elements of the sequence $\overline{\alpha}_-$, which are at the end of the latter.

Such an interpretation of the duality theorem yields an efficient routine of dual search for ⊕ and ⊖ kernels of the system W. This is due to the fact if the elements are often encountered, there exists a higher possibility of finding a ⊕ kernel at the beginning of the sequence $\overline{\alpha}_+$ as compared to finding it at the end of $\overline{\alpha}_-$; the same applies also to a ⊖ kernel in the sequence $\overline{\alpha}_-$.

The routine under construction is based on Corollaries I-IV of the duality theorem presented in Appendix II, where we also prove this theorem.

The routine of dual search for kernels described below is an application of two constructive routines, i.e., a KSR for constructing $\overline{\alpha}_+$ and a KSR for constructing $\overline{\alpha}_-$. The routine is stepwise, with two constructing stages realized at each step, i.e., a stage in which the KSR is used for constructing $\overline{\alpha}_+$ with ⊕ operations, and a stage in which the same routine is used for constructing $\overline{\alpha}_-$ with the aid of ⊖ operations on the elements of the system.

Z. At first we store two numbers: $u_0^+ = F_+(W)$ and $u_0^- = 0$. After that we perform precisely Stage 1 and 2 of the zero step of the KSR used for constructing the defining sequence $\overline{\alpha}_+$. This signifies that the set W contains an element μ_0 such that $\pi^+W(\mu_0) = \max_{\delta\in W}\pi^+W(\delta) = F_+(W)$. The threshold u_0^+ is equal to $\pi^+W(\mu_0)$, etc. By using the constructions of the zero step of KSR at the previous stage of the dual routine under construction, we obtained a set $\Gamma_1^+ \subset W$. Then we examine the set $W \setminus \Gamma_1^+$ and the credential system $\Pi^+W \setminus \Gamma_1^+$. On the set $\overline{\alpha}_+$ with the credential system $\Gamma_{j+1}^+ \subset \Gamma_j^+$ we perform a complete kernel-searching routine for the purpose of constructing a defining sequence of $\oplus$ operations only for the set $W \setminus \Gamma_{j+1}^+$. As a result, we obtain in the set $W \setminus \Gamma_1^+$ a subset F_- on which the function F_- reaches a global maximum among all the subsets of the set $W \setminus \Gamma_1^+$.

R. By applying the previous $(j-1)$ steps to the j-th step, we obtained a sequence of sets $\Gamma_0^+, \Gamma_1^+, ..., \Gamma_j^+$, and according to the construction of a defining sequence we have $\Gamma_0^+ \supset \Gamma_1^+ \supset ... \supset \Gamma_j^+$ and $\Gamma_0^+ = W$. At first we store two numbers: $u_j^+ = F_+(\Gamma_j^+)$ and $u_j^- = F_-(H^j)$. By analogy, we perform the same construction consisting of two stages of a KSR recursion step for constructing $\overline{\alpha}_+$ with the aid of $\oplus$ operations. At a given instant of such dual construction we obtained a set $\Gamma_{j+1}^+ \subset \Gamma_j^+$. Then we consider the set $W \setminus \Gamma_{j+1}^+$ and the credential system $\Pi^-W \setminus \Gamma_{j+1}^+$. In the same way as at the zero step, we perform on the set $W \setminus \Gamma_{j+1}^+$ a complete kernel-searching routine with the purpose of constructing a sequence $\overline{\alpha}_-$ only on the set $W \setminus \Gamma_{j+1}^+$. As a result we obtain in the set $W \setminus \Gamma_{j+1}^+$ a subset H^{j+1} on which the function F_- reaches a global maximum among all subsets of the set $W \setminus \Gamma_{j+1}^+$.

S. Before starting the construction of the j-th step of the routine under construction, we check the condition of a Rule of Termination of Construction Routine:

$$u_j^+ \le u_j^- . \tag{4}$$

If (4) is satisfied as a strict inequality, the construction will terminate before the j-th step. If (4) is an equality, the construction will terminate after the j-th step.

8. DEFINABLE SETS OF DUAL KERNEL-SEARCH ROUTINE

At the end of the construction process, the above routine yields a set H^j or a set H^{j+1}. It can be asserted that one of the sets is definable set or the largest kernel of the system W with respect to a collection of credential system $\{\Pi^- H \mid H \subseteq W\}$.

The assertion is based on the following. Firstly, by applying the KSR we obtained the second stage of the j-th step of a dual routine the maximal set H^{j+1} among all the subsets of the set $W \setminus \Gamma^+_{j+1}$ on which the function F_- reaches a global maximum in the system of sets of all the subsets of the set $W \setminus \Gamma^+_{j+1}$. Secondly, by virtue of Corollary 1 of the Theorem 2 (the duality theorem), it follows that, prior to the j-th step and provided that (4) is a strict inequality, the largest kernel (a definable set) will be contained in the set $W \setminus \Gamma^+_j$, or it follows from the Corollary 2 of the Theorem 2, if (4) is a equality, that the largest kernel is included in the set $W \setminus \Gamma^+_{j+1}$. Thus by comparing these two remarks we can see that either H^j or H^{j+1} is a definable set.

By virtue of Corollaries 3 and 4 of the duality theorem, it is possible to find by similar dual routine also the largest kernel $K^{\oplus}$- definable set. This assertion can be proved in the same way as the assertion about H^j and H^{j+1}; therefore this proof is not given here.

APPENDIX 1

Proof of Theorem 1. We suppose that a definable set H^*_- exists.

(Conducting the proof by contradiction) let us assume that there exists a set $L \subseteq W$, which satisfies the inequality

$$F_-(H^*) \le F_-(L). \quad \text{(A1.1)}$$

Thus two sets H^*_- and L are considered. One of the following statements holds:

1) Either $L / H^*_- \neq \varnothing$, which signifies the existence of elements in L, not belonging to H^*_-;

2) or $L \subseteq H^*_-$.

We first consider 2). By a property of definable set H^*_- there exists a defining sequence $\overline{\alpha}_-$ of elements of set W with the property b) (cf. the definition of $\overline{\alpha}_-$) such that the strict inequality $F_-(H^*) < F_-(L)$ does not hold and, consequently, only the equality holds in (A1.1). In this case, the first and the third statements of the theorem are proved. It remains only to prove the uniqueness of H^*_-, whish is done after considering 1).

Thus, let $L/H_{-}^{*} \neq \varnothing$ and let us consider set H_{t} – the smallest of those H_{i} $(i = 0,1,...,k-1)$ from the defining sequence $\overline{\alpha}_{-}$ that include the set L/H_{-}^{*}. Then the fact that H_{t} is the smallest of the indicated sets implies the following: there exists element $\lambda \in L$, such that $\lambda \in H_{t}$, but $\lambda \notin H_{t+1}$.

Below, we denote by $i(\Omega)$ the smallest of the indices of elements of defining sequence $\overline{\alpha}_{-}$ that belong to the set $\Omega \subseteq W$.

Let Γ_{p}^{-} be the last in the sequence of sets $\langle \Gamma_{j}^{-} \rangle$, whose existence is guaranteed by the sequence $\overline{\alpha}_{-}$. For indices t and $i(\Gamma_{p}^{-})$ we have the inequality $t < i(\Gamma_{p}^{-})$.

The last inequality means that in sequence of sets $\langle \Gamma_{j}^{-} \rangle$ there exists at least one set Γ_{s}^{-}, which satisfies

$$i(\Gamma_{s+1}^{-}) \geq t+1 . \qquad \text{(A1.2)}$$

Without decreasing generality, one can assume that Γ_{s}^{-} is the largest among such sets.

It has been established above that $\lambda \in H_{t}$, but $\lambda \notin H_{t+1}$. Inequality (A1.2) shows that $\Gamma_{s}^{-} \subset H_{t+1}$, since the opposite assumption $\Gamma_{s}^{-} \supseteq H_{t+1}$ leads to the conclusion that $i(\Gamma_{s}^{-}) \geq t+1$ and, consequently Γ_{s}^{-} is not the largest of the sets, for which (A1.2) holds.

Thus, it is established that $\Gamma_{s-1}^{-} \supset H_{t}$. Indeed, if $\Gamma_{s-1}^{-} \subseteq H_{t}$, then for indices $i(\Gamma_{s-1}^{-})$ and t we have $i(\Gamma_{s-1}^{-}) \geq t$.

Hence $i(\Gamma_{s-1}^{-})+1 \geq t+1$ and the inequality $i(\Gamma_{s}^{-}) \geq i(\Gamma_{s-1}^{-})+1$ implies $i(\Gamma_{s}^{-}) \geq t+1$. The last inequality once again contradicts the choice of set Γ_{s}^{-} as the largest set, which satisfies inequality (A1.2).

Thus, $\lambda \notin \Gamma_{s}^{-}$ but $\lambda \in \Gamma_{s-1}^{-}$ since $\lambda \in H_{t}$, $H_{t} \subseteq \Gamma_{s-1}^{-}$. On the basis of property a) of the defining sequence $\overline{\alpha}_{-}$, we can conclude that

$$\pi^{-}H_{t}(\lambda) < F_{-}(\Gamma_{s}) , \qquad \text{(A1.3)}$$

where $0 \leq s \leq p$.

Let us consider an arbitrary set Γ_j^- $(j = 0,1,...,p-1)$ and an element $\tau \in \Gamma_j^-$, which has the smallest index in the sequence $\overline{\alpha}_-$. In other words, set Γ_j^- starts from the element τ in sequence $\overline{\alpha}_-$. In this case, set Γ_j^- is a certain set H_i in the sequence of imbedded sets $\langle H_i \rangle$. The definition of $F_-(H)$ and the property a) of defining sequence $\overline{\alpha}_-$ implies that

$$F_-(\Gamma_j) \leq \pi^- \Gamma_j(\tau) < F_-(\Gamma_{j+1}).$$

Hence, since $\Gamma_p^- = H_-^*$ and $F_-(\Gamma_0) < F_-(\Gamma_1) < ... < F_-(\Gamma_p)$ and as a corollary we have for $j = 0,1,...,p$

$$F_-(\Gamma_j) \leq F_-(\Gamma_p) = F_-(H^*), \tag{A1.4}$$

Let $\mu \in L$ and let credential $\pi^- L(\mu)$ be minimal in the collection of credentials relative to set L. On the basis of inequalities (A1.1), (A1.3), and (A1.4) we deduce that

$$\pi^- H_t(\lambda) < \pi^- L(\mu) = F_-(L). \tag{A1.5}$$

Above, H_t was chosen so that $L \subseteq H_t$. Recalling the fundamental monotonicity property (1) for collection of credentials (the influence of elements on each other), it easy to establish that

$$\pi^- L(\lambda) \leq \pi^- H_t(\lambda). \tag{A1.6}$$

Inequalities (A.5) and (A.6) imply the inequality

$$\pi^- L(\lambda) < \pi^- L(\mu),$$

i.e., there exists in the collection of credentials relative to set L a credential, which is strictly less than the minimal credential.

A contradiction is obtained and it is proved that set L can only be a subset of H_-^* and that all sets, distinct from H_-^*, on which the global maximum is also reached, lie inside H_-^*.

It remains to prove that if a definable set H_-^* exists, then it is unique. Indeed, in consequence of what has been proved above we can only suppose that some definable set $H_-^{'}$, distinct from H_-^*, is included in H_-^*.

It is now enough to adduce arguments for definable set $H_-^{'}$ similar to those adduced above for L, considering it as definable set $H_-^{'}$; this implies that $H_-^* \subseteq H_-^{'}$. The theorem is proved. ■

APPENDIX 2

Proof of Theorem 3. Let Ω be the system of set in $P(W)$, on which function F_- reaches a global maximum, and let $K_1 \in \Omega$ and $K_2 \in \Omega$.

Since on K_1 and K_2 the function F_- reaches a global maximum, therefore we might establish the inequalities

$$F_-(K_1 \cup K_2) \le F_-(K_1), \tag{A2.1}$$

$$F_-(K_1 \cup K_2) \le F_-(K_2). \tag{A2.2}$$

We consider element $\mu \in K_1 \cup K_2$, on which the value of function F_- on set $K_1 \cup K_2$, is reached, i.e.,

$$\pi^- K_1 \cup K_2(\mu) = \min_{\alpha \in K_1 \cup K_2} \pi^- K_1 \cup K_2(\alpha).$$

If $\mu \in K_1$, then by rendering $\ominus$ actions on all those elements of set $K_1 \cup K_2$, that do not belong to K_1, we deduce from the fundamental monotonicity property of collections of credentials (1) the validity of the inequality

$$\pi^- K_1(\mu) \le \pi^- K_1 \cup K_2(\mu).$$

Since the definition of F_- implies that $F_-(K_1) \le \pi^- K_1(\mu)$ and by the choice of element μ we have $\pi^- K_1 \cup K_2(\mu) = F_-(K_1 \cup K_2)$, we therefore deduce the inequality

$$F_-(K_1) \le F_-(K_1 \cup K_2).$$

Now from the inequality (A2.1) it follows that

$$F_-(K_1) = F_-(K_1 \cup K_2).$$

If, however, it is supposed that $\mu \in K_2$, then $\ominus$ actions are rendered on elements of $K_1 \cup K_2$, not belonging to K_2; in an analogous way implementing (A2.2) we obtain the equality

$$F_-(K_2) = F_-(K_1 \cup K_2).$$

The Theorem 3 has been proved. ■

APPENDIX 3

Proof of Theorem 1. We shell prove that a sequence $\overline{\alpha}$ constructed by the KSR rules is a defining sequence for a collection of credential systems

$$\left\{\Pi^{-}H \mid H \subseteq W\right\}.$$

First of all let us recall the definition of a defining sequence of elements of the system W. We shall use the notation $\Delta_{\overline{\alpha}} = \langle H_0, H_1, ..., H_{k-1} \rangle$, where $H_0 = W$, $H_{i+1} = H_i \setminus \alpha_i$ $(i = 0,1,...,k-2)$. A sequence of elements of a set W is said to be defining with respect to a coalition of credential system $\left\{\Pi^{-}H \mid H \subseteq W\right\}$ if the sequence $\Delta_{\overline{\alpha}}$ has a subsequence of sets $\Gamma_{\overline{\alpha}} = \langle \Gamma_0, \Gamma_1, ..., \Gamma_p \rangle$, such that

a) The credential $\pi^{-}H_i(\alpha_i)$ of any element α_i of the sequence $\overline{\alpha}$ that belongs to the set Γ_j, but does not belong to the set Γ_{j+1}, is strictly smaller than the credential of an element with minimal credential with respect to the set Γ_{j+1}, i.e., $\pi^{-}H_i(\alpha_i) < F_{-}(\Gamma_{j+1})$, $j = 0,1,...,p-1$ [3];

b) the set Γ_p does not have a proper subset L such that the strict inequality $F_{-}(\Gamma_p) < F_{-}(L)$ is satisfied (the "–" symbol has been omitted; see previous footnote).

We shall consider a sequence of sets $\Delta_{\overline{\alpha}}$ and take the subsequence $\Gamma_{\overline{\alpha}}$ in the form of the sets Γ_j $(j = 0,1,...,p)$ constructed by the KSR rules. We have to prove that sets Γ_j have the required properties of a defining sequence. Assuming the contrary carries out the proof.

Let us assume that Mullat property (1971) of a defining sequence is not satisfied. This means that for any set Γ_j there exists in the sequence of elements

$$\overline{\beta}_j = \langle \beta_j(1), \beta_j(2), ..., \rangle$$

an element $\beta_j(r)$ such that

$$\pi^{-}H_{v+r}(\beta_j(r)) \geq F_{-}(\Gamma_{j+1}) = u_{j+1} \qquad \text{(A3.1)}$$

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[3] In the definition of $\overline{\alpha}_{+}$ sequence it is required that the following strict inequality be satisfied: $\pi^{+}H_i(\alpha_i) > F_{+}(\Gamma_{j+1})$, $j = 0,1,...,q-1$

Here v is the index number of the element μ_j selected in Stage 1 of the recursion step of the constructive routine of determination of $\overline{\alpha}$; in the vocabulary of notation used in Mullat (1976) we have $v = i(\Gamma_j)$.

According to the method of construction, the sequence $\overline{\beta}_j$ consists of sequences γ formed at the second stage of the j-th step of the constructive routine. Let M be a set in a sequence of sets $\Delta_{\overline{\alpha}}$ such that the first element $\alpha_{i(M)}$ of the set M in the constructed sequence $\overline{\alpha}$ is used at the second stage of the j-th step for constructing the sequence γ to which the element $\beta_j(r)$ belongs. This definition of M shows that $H_{v+r} \subseteq M$.

From the construction of the second stage of the j-th step and the principal property of monotonicity of ⊖ operations in the system we obtain the inequalities

$$\pi^{-} H_{v+r}(\beta_j(r)) \leq \pi^{-} M(\beta_j(r)) \leq \pi^{-} \Gamma_j(\mu_j) = u_j \tag{A3.2}$$

By virtue of the above method of selection of the set Γ_{j+1} from the sequence of sets $\langle \Gamma_j \rangle$ and of the properties of a fixed sequence $\overline{\beta}_j$, we obtain at the j-th step

$$u_j = \pi^{-} \Gamma_j(\mu_{j+1}) < \pi^{-} \Gamma_{j+1}(\mu_{j+1}) = u_{j+1}, \tag{A3.3}$$

where $j = 0,1,...,p-1$.

According to the rule of constructing of the sequence $\overline{\alpha}$, the function F_{-} reaches its value on the elements μ_j and μ_{j+1}. The elements μ_j and μ_{j+1} belong to the sets Γ_j and Γ_{j+1} respectively; therefore the inequalities (A.1) – (A3.3) are contradictory.

Thus our assumption is not true and Mullat Property of the defining sequence $\overline{\alpha}$ constructed by KSR rules has been proved.

Let as assume that Property b) does not hold, i.e., the last Γ_p of the sequence $\langle \Gamma_j \rangle$ contains a proper subset L such that

$$F_{-}(\Gamma_p) < F_{-}(L) . \tag{A3.4}$$

Let the element $\lambda \in L$, and suppose that it is the element with minimal ordinal number in $\overline{\alpha}$ belonging to L; moreover, let t denotes this number, i.e., $t = i(L)$, $\alpha_t = \lambda$. From the definition of t it follows that $L \subseteq H_t$.

Our analysis carried out above for the set H_{v+r} we repeat below for the set H_t. By analogy with the definition of the set M we define a set M' with the aid of the element λ and the sequence $\overline{\alpha}$.

The set M' is equated with the set of the sequence of sets $\Delta_{\overline{\alpha}}$ that begins with an element used in the formation of a set $\overline{\gamma}$ at the p-th step of the constructive routine such that $\lambda \in \overline{\gamma}$.

By analogy with derivative of (A3.2) we obtain

$$\pi^- H_t(\lambda) \leq \pi^- M'(\lambda) \geq \pi^- \Gamma_p(\mu_p) = u_p . \tag{A3.5}$$

Since $F_-(L) \leq \pi^- L(\lambda)$, it follows from (A3.4) and (A3.5) that $\pi^- H_t(\lambda) < \pi^- L(\lambda)$.

We noted above that $L \subseteq H_t$, by virtue of the monotonicity of $\ominus$ operations, it hence follows that

$$\pi^- L(\lambda) \leq \pi^- H_t(\lambda) .$$

The last two inequalities are contradictory, and hence Property b) of the defining sequence is satisfied.

Thus we have proved that the sequence $\overline{\alpha}$ constructed by the KSR rules is a defining sequence with respect to a collection of credential systems $\left\{\Pi^- H \mid H \subseteq W\right\}$, and hence it can be denoted by $\overline{\alpha}_-$, whereas the sequence $\left\langle \Gamma_j \right\rangle$ obtained by a constructive routine can be denoted by $\Gamma^-_{\overline{\alpha}_-}$.

APPENDIX 4

Proof of Duality Theorem. Below we shall show that $\Gamma^-_m \subseteq W \setminus \Gamma^+_{n+1}$, if $F_+(\Gamma^+_n) = F_-(\Gamma^-_m)$ (we omit a twice notation of + and − symbols; a promised above the + and − sign will not be used twice in notation. This rule has been applied also to Appendices 1 and 2.

Let us assume that there exists an element $\xi \in \Gamma^-_m$ and that $\xi \in \Gamma^-_{m+1}$, i.e., $\Gamma^-_m \subseteq W \setminus \Gamma^+_{n+1}$. Hence follows that we have defined a credential $\pi\Gamma^+_{n+1}(\xi)$. According to the definition of the function F_+ the following inequality is true: $\pi\Gamma^+_{n+1}(\xi) \le F(\Gamma^+_{n+1})$.

For a defining sequence $\overline{\alpha}_+$ and for any $j = 0,1,...,q-1$ we have inequalities

$$F(\Gamma^+_{n+1}) < F(\Gamma^+_n). \tag{A4.1}$$

Let us consider an element $g \in \Gamma^+_n$ with the smallest index number in $\overline{\alpha}_+$. It follows from the definition of $\overline{\alpha}_+$ that

$$\pi\Gamma^+_n(g) > F(\Gamma^+_{n+1}).$$

The choice of element g is convenient because it permits the use of Mullat Property of a defining sequence (see Appendix 1), i.e., in this case the set Γ^+_n is in the form of $H_t = \Gamma^+_n$. Since $F(\Gamma^+_n) \ge \pi\Gamma^+_n(g)$, we have proved (A4.1).

Since $\xi \in \Gamma^-_m$, it follows that we have defined a credential $\pi\Gamma^-_m(\xi)$. We have the following chain of inequalities:

$$F(\Gamma^-_m) \le \pi\Gamma^-_m(\xi) \le \pi^- W(\xi) = \pi^+ W(\xi) \le \pi\Gamma^+_n(\xi).$$

Let us recall that for any element δ of the system W under consideration, we have in a) the relation $\pi^- W(\delta) = \pi^+ W(\delta)$. The first inequality follows from the definition of the function F_-, and the second inequality from the monotonicity of $\ominus$ operations. The equality follows from the definition of the functions π^- and π^+, whereas the last inequality follows from the monotonicity of $\ominus$ operations.

By virtue of (A4.1) and of the conditions of the theorem, we have also the following chain of inequalities:

$$\pi\Gamma^+_{n+1}(\xi) \le F(\Gamma^+_{n+1}) < F(\Gamma^+_n) = F(\Gamma^-_m).$$

By supplementing this chain by the previous chain of inequalities, we hence obtain $\pi\Gamma^+_{n+1}(\xi) < \pi\Gamma^+_n(\xi)$. Since $\Gamma^+_{n+1} \subset \Gamma^+_n$, it follows from the monotonicity of $\oplus$ operations that $\pi\Gamma^+_{n+1}(\xi) < \pi\Gamma^+_{n+1}(\xi)$. The logical step used for obtaining the last inequality is valid, and therefore the assumption that $\Gamma^-_m \subseteq W \setminus \Gamma^+_{n+1}$ is untrue.

In the same way we can prove the inclusion $\Gamma_n^+ \subseteq W \setminus \Gamma_{m+1}^-$. For this purpose it suffices to change the signs of the inequalities and (whenever necessary) to replace the set Γ_{n+1}^+ by Γ_{n+1}^-, and Γ_m^- by Γ_n^+.

If condition (3) of the theorem holds, it is not necessary to use (A4.1). In this case the proof will be similar, being based on the following chain of inequalities (The proof is based on assuming the contrary, so that $\Gamma_m^- \not\subset W \setminus \Gamma_n^+$, i.e., there exists, as it were, an element $\xi \in \Gamma_m^-$ and $\xi \in \Gamma_n^+$.): $\pi\Gamma_n^+(\xi) \le F(\Gamma_n^+) < F(\Gamma_m^-) \le\le \pi\Gamma_m^-(\xi) \le \pi^- W(\xi) \le \pi\Gamma_n^+(\xi)$.

The first inequality follows from the definition of $F(\Gamma_n^+)$, the second follows from Condition (3) of the theorem, and the third from the definition of $F(\Gamma_m^-)$. The last two relations express the properties of monotonic systems. Hence in this case we have under Condition (3) also $\pi\Gamma_n^+(\xi) < \pi\Gamma_n^+(\xi)$. This completes the proof of the theorem. ■ Now follows several corollaries of Theorem 2.

Corollary 1. If for $n = \overline{0,q}$ the defining sequence is $\overline{\alpha}_+$ there exists a subset $H \subseteq W \setminus \Gamma_n^+$ such that $F_-(H) > F(\Gamma_n^+)$. Thus kernel $K^\oplus$ will belong to the set $W \setminus \Gamma_n^+$. Indeed, since a definable set is also kernel, it follows that $F_-(H) \le F(\Gamma_p^-)$, $m = 0,1,...,p$, and hence (in any case) if $m = p$, and n is selected on the basis of the condition of the corollary, then $F(\Gamma_n^+) < F(\Gamma_p^-)$. By virtue of the theorem, we therefore obtain the assertion of the corollary.

Corollary 2. If for $n = 0,1,...,q-1$ of a defining sequence $\overline{\alpha}_+$ there exists a subset $H \subseteq W \setminus \Gamma_n^+$ such that $F_-(H) = F(\Gamma_n^+)$, then the kernel $K^\oplus$ will belong to the set $W \setminus \Gamma_{n+1}^+$.

The proof follows directly from Corollary 1, by virtue of (A4.1).

Corollary 3. If for $m = 0,1,...,p$ of a defining sequence $\overline{\alpha}_-$ there exists a subset $H \subseteq W \setminus \Gamma_m^-$ such that $F_+(H) < F(\Gamma_m^-)$ then the kernel $K^\ominus$ will belong to the set $W \setminus \Gamma_m^-$. The proof of Corollary 3 is entirely similar to that of Corollary 1. It is only necessary to change the signs of the inequalities and replace the set Γ_n^+ by Γ_m^-.

Corollary 4. If for $m = 0,1,...,p-1$ of a defining sequence $\overline{\alpha}_-$ there exists a subset $H \subseteq W \setminus \Gamma_m^-$ such that $F_+(H) = F(\Gamma_m^-)$, then the kernel $K^\ominus$ will belong to the set $W \setminus \Gamma_{m+1}^-$.

LITERATURE CITED

Braverman, E.M., Dorofeyuk, A.A., Lumel'skij, V.Ya. and I.B. Muchnik. (1971). Diagonalization of connection matrix and detection of latent factors, *in: Problems of Expanding Possibilities of Automata, No. 1*, Izd. Instituta Probl. /Upravl., Moscow, pp. 42-79.

Braverman, E.M., Kiseleva, N.E., Muchnik, I.B. and S.G. Novikov. (1974). Linguistic approach to the problem of processing large massifs of information, *Avt. Tel., No. 11*, 73.

Hartigan, J. (1972). Direct Clustering of Data Matrix, *J. Amer. Statist. Assoc., 67*, No. 337, p. 123.

Kempner, Y., Mirkin, B., and I.B. Muchnik. (1997) "Monotone linkage clustering and quasi-concave set functions," Applied Mathematics Letters, 4, 19; B. Mirkin and I.B. Muchnik. (2002) "Layered Clusters of Tightness Set Functions," Applied Mathematics Letters, v. 15, issue no. 2, pp. 147-151; see also, A. V. Genkin and I.B. Muchnik. (Moscow, Boston, 1993) "Fixed Approach to Clustering, Journal of Classification," Springer, 10, pp. 219-240; and latest connection, Kempner, Y. and V.E. Levit. (2003) "Correspondence between two antimatroid algorithmic characterizations," Dept. of Computer Science, Holon Academic Institute of Technology, July, Israel.

McCormick, W., Schweitzer, P. and T. White. (1972). Problem decomposition and data recognition by clustering technique, *J. Oper. Res., 20*, No. 5, p. 993.

Muchnik, I.B., and L. Shvartser. (1990) "Maximization of generalized characteristics of functions of monotone systems," Automation and Remote Control, 51, 1562-1572,

Muchnik, I.B. (1974). Analysis of structure of extremal graphs, *Avt. Telem., No. 9*, p. 62.

Mullat, J.E. a). (1971). On a Maximum Principle for Certain Functions of Sets, in "Notes on Data Processing and Functional Analysis", Proceedings of the Tallinn Polytechnic Institute Series A, No. 313, Tallinn Polytechnic Institute, pp. 37–44, (in Russian), http://www.datalaundering.com/download/RG-193-201-Modular.pdf;
b). (1976). Extremal Subsystems of Monotonic Systems, 1, Avtomatika & Telemekhanika, No. 5, pp. 130–139 (in Russian);
c). (1977). Data Structure Opening Method: Methodological Guide, http://www.datalaundering.com/download/RG-211-244-method-guide.pdf.

Налимов В.В. and З.М. Мульченко. (1969). Science-metry (in Russian), *Nauka*.

Ojaveer, E., Mullat, J. E. and L.Võhandu, (1975). A study of infraspecific groups of the Baltic east coast autumn herring by two new methods based on cluster analysis, *in: Proceedings of Estonian Studies According to International Biology Program VI* (in Russian), Isd. Akad. Nauk ESSR, Tartu.

Shorin O.A. (2016). Optimizing the size of the ring antenna and the rule formation of territorial clusters for cellular network McWILL, "Elektrosvjaz," No.1, pp. 22-27,, Method of "Adaptive Distribution of Bandwidth Resource", Russian Federation, Federal Service for Intellectual Property, RU 2 640 030 C1, Application 2017112131, in Russian,

Терентьев, П.В. (1959). Метод крелляционных плеяд, *Vestn. LGU, No. 9*, p. 137 .

Trybulets, A.O. (1970). Document-graph method in the classification of sciences, in*: Application of Universal Computers in the Work of Data Organs* (*Symposium Proceedings*, Moscow, June, 1967), Izd. VINITI.

Võhandu, L.K. (1964). "Investigation of multi-sign biological systems," *in: Application of Mathematical Methods in Biology, Vol. III*, Isd. LGU, pp. 19-22.

Zahn, T. (1971). "Graph-theoretical methods for detecting and describing gestalt clusters," *IEEE Trans. Comput., C-20*, No. 1, p. 68.

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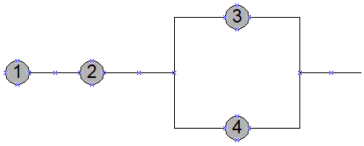
*i The name “Monotonic System” at that moment in the past was the best match for our scheme. However, this name “Monotone System” was already occupied in “Reliability Theory” unknown to the author. Below we reproduce a fragment of a “monotone system” concept different from ours in lines of Sheldon M. Ross “Introduction to Probability Models”, Fourth Ed., Academic Press, Inc., pp. 406-407.

Example

(A four-Component Structure):



Consider a system consisting of four components, and suppose that the system functions if and only if components 1 and 2 both function and at least one of components 3 and 4 function. Its structure function is given by

$$\phi(x) = x_1 \cdot x_2 \cdot \max(x_3, x_4).$$

Pictorially, the system is shown in Figure. A useful identity, easily checked, is that for binary variables, (a binary variable is one which assumes either the value 0 or 1) x_i, $i = 1,\ldots,n$,

$\max(x_1,\ldots,x_n) = 1 - \prod_{i=1}(1 - x_i)$. When $n = 2$, this yields

$$\max(x_1, x_2) = 1 - (1 - x_1)\cdot(1 - x_2) = x_1 + x_2 - x_1 \cdot x_2 .$$

Hence, the structure function in the above example may be written as

$$\phi(x) = x_1 \cdot x_2 \cdot (x_3 + x_4 - x_3 \cdot x_4)$$

It is natural to assume that replacing a failed component by a functioning one never lead to a deterioration of the system. In other words, it is natural to assume that the structure function $\phi(x)$ is an increasing function of x, that is, if $x_i \le y_i$, $i = 1,\ldots,n$, then $\phi(x) \le \phi(y)$. Such an assumption shall be made in this chapter and the system will be called *monotone*.